D103. Coupled pendulums – small angle approximation

***Aim**: Examination of oscillations of two coupled physical pendulums*
- *examination of relation between the frequency of oscillations of a simple pendulum and its moment of inertia, determination of the centre of gravity.*
- *examination of motion of two identical pendulums coupled through a spring*
- *examination of two different pendulums coupled through a spring.*

***Problems**:*
- *equation of motion for a physical pendulum, frequency of free oscillations, relation between the moment of inertia of a body and the frequency of its free oscillations.*
- *normal oscillations, equations describing normal oscillations for a system of two coupled pendulums, sum of normal oscillations, beats.*

Experimental tools needed:
- *two physical pendulums to which weights can be attached. A spring for coupling the two pendulums.*
- *potentiometers connected to each pendulum at the pivot points, that return voltage as a function of angular position of each pendulum.*
- *interface permitting measurement of voltage on the potentiometer by a computer.*
- *LabView environment through which it is possible to connect with the measuring interface and perform measurements. The voltage at particular potentiometers is read via the function Odczyt.vi or in the more advanced form – via the serial communication with the measuring interface.*

1. Free oscillations of physical pendulum.

Most often studied is a mathematical pendulum which is a simplified version of a physical one in which the bob performing oscillations is assumed to be a point of mass and the rod (or cord) on which the bob swings is assumed to have no mass. In this experiment the physical pendulums are used, see Fig. 1.

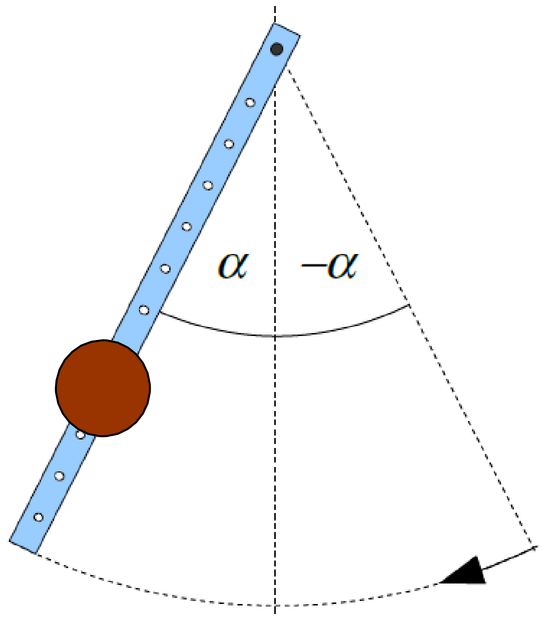

Figure.1. A physical pendulum used in the experiment.

The difference between a physical and mathematical pendulums is that the physical one is a rigid body able to rotate about the pivot point which is other than the centre of gravity of this body. To describe its motion it is necessary to take into account the moment of inertia $I$of the pendulum and position of its centre of gravity with respect to the pivot point. The Newton's second law of motion for rotations takes the form:

$$M = \frac{d\vec{L}}{dt} \tag{1}$$

where M is the net moment of force acting on the pendulum, while L is the moment of momentum related to the moment of force applied to the pendulum. L is related to the moment of inertia I of the rotating body through the equation:

$$\vec{L} = I\vec{\omega} \tag{2}$$

where ω is the vector of angular velocity with the same direction and sense as M. The above equations can be applied to physical as well as mathematical pendulums. The difference appears in determination of *M*. For both types of pendulum, the only source of the moment of force is the gravity. For the mathematical pendulum the moment of force is:

$$\left|\vec{M}\right| = \left|\vec{r} \times \vec{F}_G\right| = mgl\sin(-\alpha) = -mgl\sin(\alpha) \tag{3}$$

where *m* is the mass of the body and *l* is the distance from the pivot point or the length of the pendulum. The sign minus means that, similarly as for a spring, the moment of force acting on the pendulum and related to gravity is directed so that to counteract the displacements and return the pendulum towards the equilibrium position.

In the physical pendulum, the gravitation force acting on each fragment of the rigid body is the same, while the distances between each fragment and the pivot point are different. Therefore, the moment of force acting on the pendulum must be determined as a net moment of all moments acting on infinitesimally small fragments $m_i$of the body. The net moment of force is equal to the sum of all component moments

$$\left|\vec{M}_g\right| = \int \left|d\vec{M}_i\right| = \int gr_i \sin(-\alpha) dm_i = -g\sin\alpha \int r_i dm_i = -D\sin\alpha \tag{4}$$

where ri is the distance of mass dmi from the pivot point, D is the directing moment of the pendulum. Taking into account that $\omega = \frac{d\alpha}{dt}$, and assuming small amplitude α, which permits assuming that sin(α) ≈ α, and combining equations (1), (2) and (4) we get the equation of motion for a single physical pendulum:

$$\frac{d^2\alpha}{dt^2} + \frac{D}{I}\alpha = 0 \tag{5}$$

Please note that the amplitude 〈is the only parameter varying in time that is needed for description of motion of a given pendulum. The solution to the differential equation (5) takes the form:

$$\alpha = \alpha_0 \cos(\omega_0 t + \varphi) \tag{6}$$

where $\omega_0 = \sqrt{\frac{D_p}{I}}$ is the frequency of free oscillations of the pendulum, α_0 is the amplitude that can be found from the angular frequency ($\frac{d\alpha}{dt}$) after having assumed appropriate initial conditions (note that angular frequency differs from the pendulum eigenfrequency). φ is the initial phase of the oscillation.

Determination of the moment of inertia of the pendulum which is a rod with holes, presented in Fig. 1, is not an easy task, however, in approximation we can neglect the holes and find I for a rectangular rod of the dimensions a, b and c. Moment of inertia $I_ś$ of such a rod is measured with respect to the axis of rotation passing through its centre of mass and parallel to the side c of the rod and can be found from the equation:

$$I_{ś} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{0}^{c} (x^2 + y^2) \rho dx dy dz = \frac{1}{12} m_p (a^2 + b^2) \qquad (7)$$

where ρ is the density of the material of the rod. In our experiment the rod oscillates about the pivot point at a distance r from the centre of the rod mass. The centre of mass is in the middle of the uniform rod, so r can be measured and using the Steiner theorem the moment of inertia of the rod I_p with respect to the axis passing through the pivot point (black dot in Fig. 1).

$$I_p = I_{ś} + mr^2 \qquad (8)$$

A cylinder weight can be attached to the rod at different sites. If it is attached, then also its moment of inertia should be added to the total moment of inertia of the pendulum,

$$\begin{aligned} I &= I_p + I_w, \text{ gdzie} \\ I_w &= \frac{1}{2} m_w R_w^2 + m_w d_w^2. \end{aligned} \qquad (9)$$

In the above equation for the total moment of inertia I of the rod with weight, m_w is the mass of the cylinder, R_w is its radius, and d_w is the distance between the pivot point and the centre of mass of the cylinder.

Tasks

- *Using the program provided, record the time dependence of the amplitude of one of the pendulums with the cylinder weight at the bottom end of the rod. Set the pendulum in motion with the maximum possible amplitude and let it stop. With the help of cursors read off the frequency f of the pendulum oscillations at different amplitudes α_0 and make the plot of f(α_0).Find out at which amplitude α_g the approximation of small amplitudes ensures the acceptable accuracy. The following measurements make at the selected or smaller amplitude a_g.*
- *Changing the positions of the cylinders on the rods, on one rod from bottom up and on the other from top down, find the frequencies of free oscillations of the pendulums as a function of d_W.*
- *Measure the dimensions of the rods, positions of the holes in which cylinders can be attached and the dimensions and mass of the cylinder weights. Knowing that the directing moment of the pendulum with a cylinder weight D is a sum of the directing moments of the rod and the cylinder, calculate the frequencies of free oscillations of the pendulum for three different positions of the weight and compare the results with experimental data.*

2. Normal oscillations.

If the two pendulums are connected by a spring, in the way presented in Fig. 2, we get a system of two coupled pendulums.

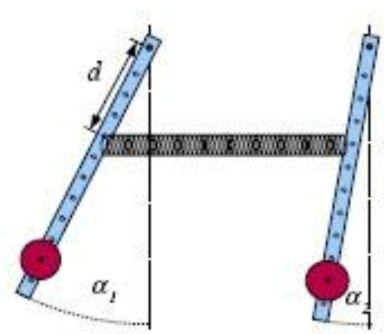

Figure 2. A system of two coupled pendulums.

This system has two degrees of freedom which are the amplitudes of the two pendulums α_1 and α_2 (in Fig. 2 the angles have positive signs according to the convention assumed in Fig. 1). It should be remembered that the spring must be attached in such a way that it would be in equilibrium for the two pendulums in equilibrium. The system is called a sympathetic pendulum and it behaves in the same way as a system of two balls connected through a spring with each other and with the neighbouring walls, which is the simplest model of two atoms linked by a bond inside a molecule or a crystal. Although the motion of a sympathetic pendulum is complex, it can always be described as a superposition of two independent harmonic motions (because of the two degrees of freedom). These motions, known as the normal oscillations, are independent and involve simultaneous oscillations of both pendulums in one or the other type of motion. The normal oscillation is the one in which all coordinates (in our experiment α_1 and α_2) change with the same frequency and with the same or exactly opposite phase. The equation describing the normal oscillations is:

$$\alpha_1 = A\cos(\omega t + \delta)\,,\ \alpha_2 = B\cos(\omega t + \delta)\,. \tag{10}$$

The normal oscillations have strictly defined frequency and amplitude. Before we find them, we have to define the equation of motion for the sympathetic pendulum, which is different from eq. (5) because of the presence of a spring. In the system of pendulums from Fig. 2, besides the gravitational force also the restoring force, acting in the horizontal direction, contributes to the moment of force M (4). The force depends on the state of the spring (extension/ contraction Δx) and the elasticity constant k, $\vec{F}_s = -k\Delta\vec{x}$. Let's consider one of the two pendulums. A change in the length of the spring relative to its length at equilibrium depends on the amplitudes of the two pendulums in the following way:

$$\Delta x = d\sin\alpha_2 - d\sin\alpha_1 \tag{11}$$

The moment of the restoring force is:

$$M_s = dF_s\sin(\frac{\pi}{2} - \alpha_1) = -kd^2(\sin\alpha_2 - \sin\alpha_1)\cos\alpha_1 \tag{12}$$

and the moment of the gravitation force according to eq.(4) is

$$M_G = -mgr\sin\alpha_1 \tag{13}$$

The net moment of force is thus equal to the sum of M_S and M_G. Taking into account the approximation of small amplitude, $\sin(\alpha_1) \approx \alpha_1, \sin(\alpha_2) \approx \alpha_2, \cos(\alpha_1) \approx 1, \cos(\alpha_2) \approx 1$, we can derive the equations of motion for both pendulums, as we have done earlier. We get:

$$\frac{d^2\alpha_1}{dt^2} + (\omega_0^2 + H)\alpha_1 - H\alpha_2 = 0 \tag{14}$$

$$\frac{d^2\alpha_2}{dt^2} + (\omega_0^2 + H)\alpha_2 - H\alpha_1 = 0 \tag{15}$$

The frequency of free oscillations ω_0 is assumed to be the same (the pendulums are loaded with the same weights), while $H = \frac{kd^2}{I}$. Equations (14) and (15) describe the oscillations of both pendulums which can be decomposed into normal oscillations (10). In order to determine the frequencies and amplitudes of normal oscillations, the equation for α_1 and α_2 (10) is substituted to (14) and (15) to get the set of equations

$$\begin{aligned} &\left[\left(\omega_0^2 - \omega^2 + H\right)A - HB\right]\sin\left(\omega t + \delta\right) = 0 \\ &\left[-HA + \left(\omega_0^2 - \omega^2 + H\right)B\right]\sin\left(\omega t + \delta\right) = 0 \end{aligned} \tag{16}$$

This set of equations (16) must be satisfied for each moment, so

$$\begin{aligned} &\left(\omega_0^2 - \omega^2 + H\right)A - HB = 0 \\ &-HA + \left(\omega_0^2 - \omega^2 + H\right)B = 0 \end{aligned} \tag{17}$$

Set (17) has a non-trivial solution when the determinant of the matrix of coefficients is zero.

$$\begin{vmatrix} \omega_0^2 - \omega^2 + H & -H \\ -H & \omega_0^2 - \omega^2 + H \end{vmatrix} = 0 \tag{18}$$

Equation (18) leads to a quadratic equation:

$$\omega^4 - 2(\omega^2 + H)\omega^2 + \omega_0^4 + 4H\omega_0^2 = 0 \tag{19}$$

whose solutions are:

$$\omega_1 = \omega_0 \tag{20}$$

$$\omega_2 = \sqrt{\omega_0^2 + 2H} \tag{21}$$

Substituting the frequencies (20) and (21) to equations (17) we realise that

$$\text{for } \omega_1 = \omega_0 \quad B_1 = A_1, \tag{22}$$

$$\text{for } \omega_2 = \sqrt{\omega_0^2 + 2H} \quad B_2 = -A_2 \tag{23}$$

Equation (22) means that if the two pendulums oscillate in the same frequency equal to the frequency of free oscillations, then the phases of their motions are the same and the amplitudes of their oscillations are the same. Equation (23) describes the situation when the frequencies of the pendulums oscillations are different from that of free oscillations, amplitudes of the two pendulums are the same but the phases are the opposite. As mentioned before, any oscillations of coupled pendulums can be described as a superposition of normal oscillations. Using equations (10), (22) and (23) we get

$$\alpha_1 = A_1 \cos(\omega_1 t + \delta_1) + A_2 \cos(\omega_2 t + \delta_2) \tag{24}$$

$$\alpha_2 = A_1 \cos(\omega_1 t + \delta_1) - A_2 \cos(\omega_2 t + \delta_2) \tag{25}$$

The normal oscillations of pendulums can be obtained by choosing proper initial conditions. For the normal oscillations type I, eq. (22), the two pendulums should be set in motion with the same initial amplitudes. For the normal type II, eq.(23), the two pendulums should be set in motion with the same initial amplitudes but in the opposite directions.

***Tasks**:*

- *Using the available program, measure the time dependencies of two coupled pendulums of known frequencies of free oscillations, moments of inertia and directing moment of the pendulum, for a few different coupling constants determined by the distance between the spring and the axis of rotation. .*
- *Read off the frequencies of normal oscillations of the sympathetic pendulum and compare with the calculated values.*
- *In order to do this you have to determine experimentally the elasticity constant of the spring, k. You can do this with the use of a weight of a known mass, thread or a piece of wire for fastening of the weight to the spring and a ruler.*

4. Beats .

Any oscillation can be described knowing its initial conditions. Let's assume that only one pendulum is set in motion with the initial amplitude of $\alpha_1(0) = \alpha_0$. Let at t =0, the initial phase be $\delta_1 = \delta_2 = 0$, α_2=0, and the initial velocities of the two pendulums be zero, $\left.\frac{d\alpha_1}{dt}\right|_{(t=0)} = 0$, $\left.\frac{d\alpha_2}{dt}\right|_{(t=0)} = 0$. Substituting these values to equations (24) and (25) gives:

$$\alpha_1(0) = \alpha_0 = A_1 + A_2, \tag{26}$$

$$\alpha_2(0) = 0 = A_1 - A_2, \tag{27}$$

Having added or subtracted equations (26) and (27) by sides, we get $A_1 = A_2 = \frac{\alpha_0}{2}$. Finally, inset these values to equations (24) and (25), which gives the following relations:

$$\alpha_1 = \frac{\alpha_0}{2}\left(\cos\omega_1 t + \cos\omega_2 t\right) = \alpha_0 \cos\frac{\omega_1 - \omega_2}{2}\cos\frac{\omega_1 + \omega_2}{2} \tag{28}$$

$$\alpha_1 = \frac{\alpha_0}{2}\left(\cos\omega_1 t - \cos\omega_2 t\right) = -\alpha_0 \sin\frac{\omega_1 - \omega_2}{2}\,\mathrm{t}\,\sin\frac{\omega_1 + \omega_2}{2}t \tag{29}$$

Substituting

$$\frac{\omega_1 + \omega_2}{2} = \omega_{śr} \quad , \quad \frac{\omega_1 - \omega_2}{2} = \omega_{\mathrm{mod}} \tag{30}$$

we get

$$\alpha_1 = \alpha_0 \cos\left(\omega_{\mathrm{mod}} t\right)\cos\left(\omega_{śr} t\right) = A_{\mathrm{mod}}(t)\cos\left(\omega_{śr} t\right) \;, \tag{31}$$

The two pendulums oscillate with the same average frequency $\omega_{śr}$, and their amplitudes are modulated with the frequency ω_{mod}, and their phases are the opposite. This phenomenon is called beats. The single cycle in which the maximum amplitude of one pendulum is passed to the other pendulum which goes from zero amplitude to its maximum amplitude and then the energy is transferred back to the first pendulum until it reaches the maximum amplitude again, is called a single beat. The time period of completion of this cycle is called the period of beats and its reciprocal is the frequency of beats.

Tasks

- *Using the available program, measure the time dependence of beats a system of two coupled pendulums after one of them has been put in motion.*
- *With the help of cursors, find the frequency of beats and the average frequency of oscillations of the two pendulums.*
- *Set the system of pendulums in motion so that they would perform the first and then the second normal oscillations. Record the time dependencies of the amplitudes of the two pendulums. With the use of the available program find the frequency of each normal oscillation.*
- *Calculate the frequency of beats and the average frequency of oscillations of both pendulums using the measured frequencies of both normal modes. Compare the results.*

5. List of references.

1. Frank S. Crawford Jr. “Fale”, wyd.IIPWN, Warszawa1975 – available at the Faculty of Physics library.
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