

**The guide to experiments  
at Basic Physics Laboratory – mechanics.**

*Tutor:*

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***<http://ppef.amu.edu.pl>***

## **General rules:**

1. To get credit for laboratory classes in the 'Physics' module, the student must obtain passing grade for at least 9 out of 10 experiments.
2. An experiment that has been missed will be rated as zero added in the calculation of the grade average.
3. Performance of an experiment consists of:
  - Theoretical preparation to an experiment to be performed on a given day. (*'A Guide to Experiments at Basic Physics Laboratory'*)
  - a) ...' and a recommended handbook *Physics* by Douglas C. Giancoli) – **oral or written introductory test.**
  - b) Making measurements and writing the results in a protocol – admission to carry out experiments **depends on the assessment of theoretical preparation.**
  - c) Analysis of the measurement results in a paper protocol in accordance with '*A Guide...*' and lecturers recommendations.

**The analysis should be handed over to the instructor during the same class.**

It is necessary to obtain a positive grade for each of the aforementioned points to receive a credit for an experiment.

**Students are obliged to come to classes on time. Those who will show up 15 minutes after the commencement of a class will not be allowed in the Laboratory.**

## **Health and safety requirements in the Basic Physics Laboratory:**

1. You may enter the Laboratory only together with the instructor / teacher/ and after collecting appropriate apparatus (as stated in '*A Guide ...*').
2. Electrical devices can be connected to the mains only in the presence of the instructor.
3. A person who damaged or destroyed the Laboratory equipment is financially responsible for it.
4. Disconnection of measurement system is carried out in the presence of the instructor .
5. After completion of measurements, the measurement station must be tidied up and the equipment must be returned to the lab assistants' room.
6. Eating and drinking is forbidden in the Laboratory.
7. All outerwear and luggage must be left in the cloakroom.

## List of experiences

1. **F100.** Determination of density by a pycnometer.
2. **F102.** Determination of temperature dependence of a viscosity coefficient of a liquid by a rotational viscometer.
3. **F108.** Measurement of aerodynamic resistance for bodies of different shapes.
4. **F103.** Elastic collisions on an air track.
5. **F110.** Verification of the equation for rigid body rotation.
6. **F109.** Vibrations of coupled pendulums.
7. **M102** Energy transformations in the movement of pendulum (FILAMI)
8. **M104** Free fall and slide on an inclined plane (FILAMI)
9. **D103** Coupled physical pendulums (FILAMI)
- 10.**D101** Properties of sound waves. Sound velocity and beats (FILAMI)

## **Algorithm for evaluation of measurement uncertainties, the wps<sup>1</sup> principle**

(after B. Piątek, Physics at school, 1/1976)

1. Find the formula relating the measured physical quantity with other physical quantities or with results of simple measurements of the same quantity:

$$y = f(a, b, c \dots).$$

2. Find out if the measurements of the quantities  $a, b, c, \dots$  are simple measurements or a sum or difference of a few simple measurements of the same quantity or a product or ratio of results of simple measurements by a dimensionless constant, or a combination of the above.

3. On the basis of the accuracies of the tools or instruments used for simple measurements of particular quantities and assuming the worst possible scenario<sup>1</sup> (wps principle) evaluate the absolute uncertainties  $\Delta a, \Delta b, \Delta c, \dots$  of the quantities  $a, b, c, \dots$ .

4. Calculate the value of the measured quantity  $y$ , following the principles of approximations.

5. Find the highest possible values of  $a, b, c, \dots$  measured as

$$a + \Delta a; b + \Delta b; c + \Delta c \dots,$$

and the lowest possible values  $a - \Delta a; b - \Delta b; c - \Delta c \dots$

6. On the basis of the worst possible scenario principle calculate the highest possible value of the quantity measures by a given method and with the use of given measuring instruments:

$$y_{\max} = f(a \pm \Delta a; b \pm \Delta b; c \pm \Delta c),$$

calculate the smallest possible value of a given quantity

$$y_{\min} = f(a \pm \Delta a; b \pm \Delta b; c \pm \Delta c).$$

Depending on the type of the formula, that is e.g. on whether a given quantity is in the numerator or denominator, in order to get the maximum value we add the uncertainties (when in the numerator) or subtract the uncertainties. If the formula is a combination of fractions and sums, one should analyze where the summation and where the subtraction should be made to get the maximum value.

7. Calculate  $\Delta y_+ = |y_{\max} - y|$  and  $\Delta y_- = |y - y_{\min}|$

8. If  $\Delta y_+ \neq \Delta y_-$ , then take the greater of these two differences as the maximum uncertainty  $\Delta y$  of the measured quantity  $y$ .

9. Write the final result of the measurement as

$$y \pm \Delta y$$

10. Express the measurement uncertainty in percentage ( $\Delta y/y \times 100\%$ )

<sup>1</sup> The worst possible scenario principle, wps: on evaluating the uncertainties of measurements, assume the worst possible scenario from the point of view of the measurement accuracy.

## Determination of solid bodies density with the use of a pycnometer

Aim: Determine the metal density

Pycnometer is a device for measurement of the volume of solids. It is a glass flask tightly closed with a cork with a capillary hole. Fill the pycnometer with water, then place in the flask solid bodies whose density you wish to determine. The volume of water that will flow out of the flask is equal to the volume of the solid bodies placed in the flask. The density of the solid body or bodies placed in the flask is calculated from the formula:

$$(1) \quad \rho_x = \frac{m_x}{v_x}.$$

The mass of the solid body  $m_x$  is measured using the laboratory balance. The volume of the solid body  $v_x$  is calculated from the formula:

$$(2) \quad v_x = \frac{(m_x + m_y) - m_z}{\rho_{H_2O}},$$

where

$\rho_{H_2O}$  - the density of water,

$m_x$  - the weighted mass of the solid body

$m_y$  - the mass of the pycnometer filled with water

$m_z$  - the mass of the pycnometer filled with water and the pieces of metal.

The volume of water that spilled out from the pycnometer after placing in it the pieces of metal is equal to the volume of the metal pieces:

$$(3) \quad v_{H_2O} = v_x.$$

The mass of the spilled water  $m_{H_2O}$  is equal to the mass difference:

$$(4) \quad m_{H_2O} = (m_x + m_y) - m_z,$$

Therefore,

$$v_x = v_{H_2O} = \frac{m_{H_2O}}{\rho_{H_2O}} = \frac{m_x + m_y - m_z}{\rho_{H_2O}}.$$

The metal density can be calculated from

$$(5) \quad \rho_x = \rho_{H_2O} \frac{m_x}{m_x + m_y - m_z},$$

And the measurement uncertainty is

$$(6) \quad \Delta\rho_x = \rho_{H_2O} \left( \left| \frac{(m_x - m_z + m_y) - m_x}{(m_x - m_z + m_y)^2} \right| \Delta m_x + \left| \frac{m_x}{(m_x - m_z + m_y)^2} \right| \Delta m_z + \left| \frac{m_x}{(m_x - m_z + m_y)^2} \right| \Delta m_y \right)$$

### Measurements

Weigh dry pieces of metal ( $m_x$ ), pycnometer filled with water ( $m_y$ ) and pycnometer filled with water and metal pieces ( $m_z$ )

### Calculations

Remember to appropriately change the units. Results of measurements substitute to formula (5). The uncertainty of measurements is calculated from eq. (6). Compare the final result with the actual density of copper and aluminum given in tables.

$$\rho_{Cu}^T = 8,89 \cdot 10^3 \frac{kg}{m^3}, \quad \rho_{Al}^T = 2,699 \cdot 10^3 \frac{kg}{m^3}$$

Apply the criterion of agreement:  $|\rho^T - \rho_x| \leq \Delta\rho_x$ .

If the above inequality is met, your result is in agreement with the actual value, otherwise analyze the possible sources of error.

Density of water at temperatures from the range 15-25 °C in  $\frac{kg}{m^3}$

Temp. w °C	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9
15	999,099	999,084	999,069	999,054	999,038	999,023	999,007	998,991	998,975	998,959
16	998,943	998,926	998,910	998,893	998,877	998,860	998,843	998,826	998,809	998,792
17	998,774	998,757	998,739	998,722	998,704	998,686	998,668	998,650	998,632	998,613
18	998,595	998,576	998,558	998,539	998,520	998,501	998,482	998,463	998,444	998,424
19	998,405	998,385	998,365	998,345	998,325	998,305	998,285	998,265	998,244	998,224
20	998,203	998,183	998,162	998,141	998,120	998,099	998,078	998,056	998,035	998,013
21	997,992	997,970	997,948	997,926	997,904	997,882	997,860	997,837	997,815	997,792
22	997,770	997,747	997,724	997,701	997,678	997,655	997,632	997,608	997,585	997,561
23	997,538	997,514	997,490	997,466	997,442	997,418	997,394	997,369	997,345	997,320
24	997,296	997,271	997,246	997,221	997,196	997,171	997,146	997,120	997,095	997,069
25	997,044	997,018	996,992	996,967	996,941	996,914	996,888	996,862	996,836	996,809

## Experiment F102

### Temperature dependence of liquid's viscosity with the help of a rotational viscometer

Viscosity or the internal friction is the property characterizing gases, liquids and some solids. It describes the resistance against flowing stimulated by external forces and is a consequence of intermolecular interactions. Viscosity is measured by the coefficient of viscosity, defined as the strength of tangent force  $F$  that needs to be applied to the unit area of the shifting layer  $S$  in order to maintain in this layer the laminar flow with a constant gradient of frequency  $G$  equal to 1. For the uniform shape deformation for which  $G = const$ , the viscosity  $\eta$  is defined by the formula:

$$\eta = \frac{\tau}{G}, \quad (1)$$

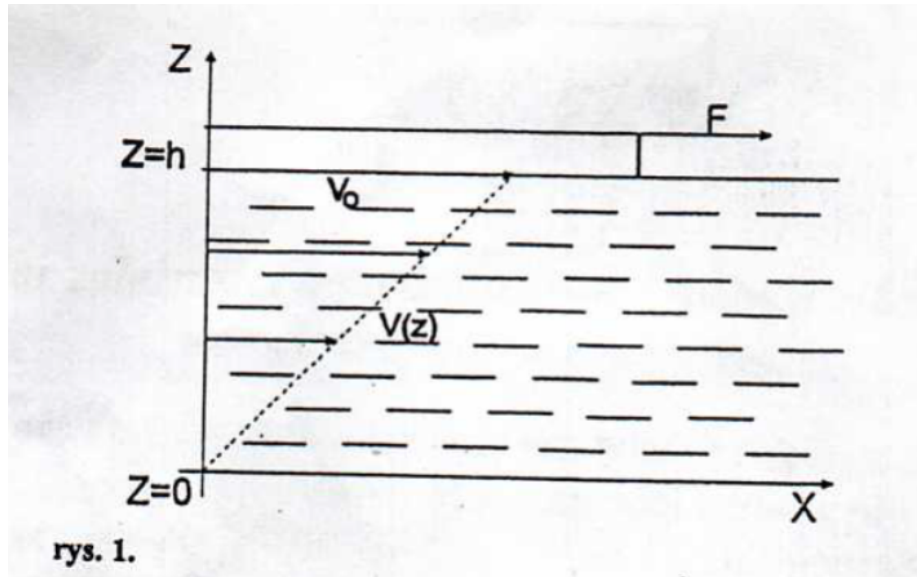
where

$$\tau = \frac{F}{S} \quad (2)$$

is the tangent stress and

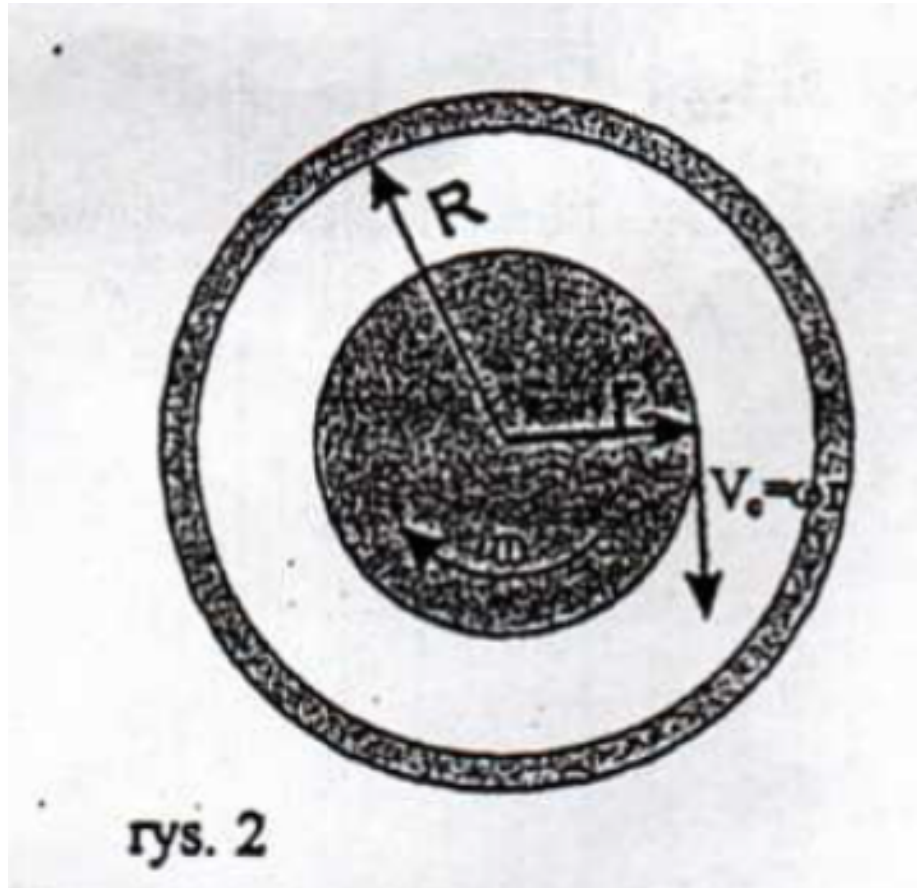
$$G = \frac{dV}{dz} = \frac{V_0}{h} \quad (3)$$

is the gradient of velocity  $V$ , the values of  $V_0$  and  $h$  are defined in Fig. 1. The SI unit of viscosity is  $[kg\ m^{-1}\ s^{-1}] = [N\ S\ m^{-2}]$  while the CGS unit is  $[p\ u\ a\ z =\ d\ y\ n\ a\ s\ c\ m^{-2} = 0.1\ N\ s\ m^{-2}]$



Viscosity is measured by viscometers. One of them is the so-called rotational viscometer. The main elements of this device is a system of two co-axial cylinders. The external one of radius  $R$  is immobilized, while the internal one of the

radius  $r$  rotates at a constant angular velocity  $\omega$ . The liquid studied is placed between the cylinders and because of the rotations of the internal cylinder a gradient of velocity is generated in it.



The rotation is transmitted from the rotor to the moving cylinder through a spiral spring. The angle of rotation of this spring is proportional to the moment of force  $M$  which should be applied to the cylinder to maintain a constant angular frequency  $\omega$ . This moment is also the moment of the force of viscosity:

$$M = rF \quad (4)$$

The deviation of the spring from the equilibrium position is measured by a resistor potentiometer in the bridge circuit. The deviation of the amperemeter indicator in the diagonal arm of the bridge is proportional to the moment of force of viscosity:

$$M = k\alpha \quad (5)$$



As the tangent stress is described by eq. (2), then on the basis of eq. (4) and (5) and the formula for the side area of the cylinder:

$$S = 2\pi rl \quad (6)$$

we get:

$$\tau = Z\alpha \quad (7)$$

Where  $Z$  is a constant defined as follows:

$$Z = \frac{k}{2\pi r^2 l} \quad (8)$$

The value of this constant is given in the user manual for the viscometer. The value of the gradient  $G$  is also given in the user manual:

$$G = \frac{r}{R-r}\omega \quad (9)$$

For different angular velocity that is for the right number of the gear (from 1 to 12) and the right symbol of type of transmission (a or b).

Thus, the viscosity can be found from the simple formula

$$\eta = \frac{Z}{G'}\alpha \quad (10)$$

where  $G' = Gv/50$  is the value of the gradient corrected for a given frequency in Hz, as the velocity of the synchronized engine used in the viscometer depends on the voltage frequency and the value of gradient  $G$  in the table is given for  $v=50\text{Hz}$ .

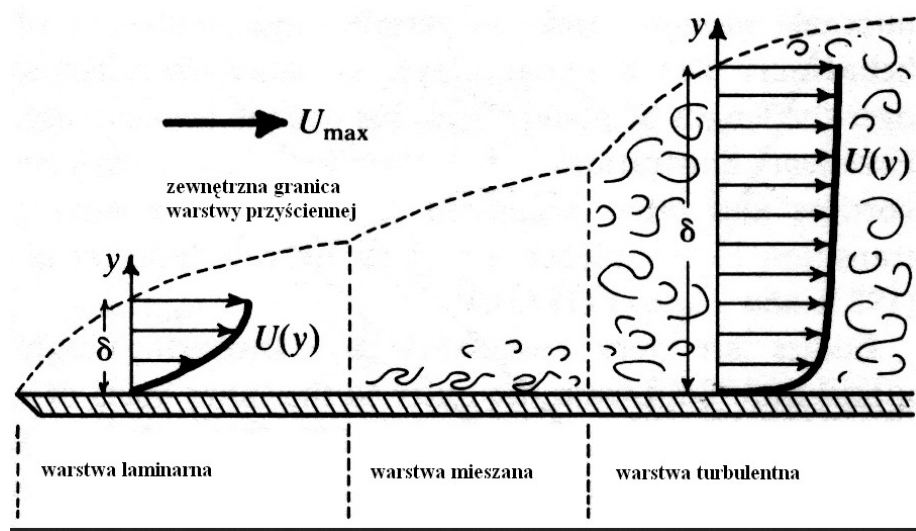
Viscosity of a liquid strongly depends on temperature and determination of the character of this dependence is the subject of the experiment. The temperature of the liquid studied is controlled and stabilized by a thermostat. The thermostat system comprises a water bath surrounding the cylinders and maintaining their constant temperature.

# Experiment F108

## Drag force

Any object moving in a fluid (as a fluid we understand a liquid or gas) is subject to a force acting in the direction opposite to the direction of the object motion, called the drag force, air resistance or fluid resistance,  $P$ .

Drag depends on the type of fluid (its density and type of fluid flow) and the object velocity. The fluid flow can be laminar or turbulent. **Turbulent flow** – is characterized by chaotic changes in pressure and velocity, the fluid particles move on collision trajectories, often form vortices, move back and forward, make collisions, which leads to the fluid mixing. **Laminar flow** – in this type of flow the fluid flows in parallel layers with no disruption between them. This type of flow takes place at low flow velocities. The upper limit of flow velocity at which laminar flow changes into turbulent flow can be calculated for a given fluid and flow conditions.



The problem of an object moving in fluids has been considered by Newton and Stokes. They have derived formulae relating the drag force  $P$  to the object velocity. Stokes considered the laminar flow of a ball in the fluid. Newton did not assume the laminar flow nor the ball-shape of the object and derived the analogous formula for the turbulent flow.

The drag force according to Newton is:

$$P = CS \frac{\rho v^2}{2} \quad (1)$$

where

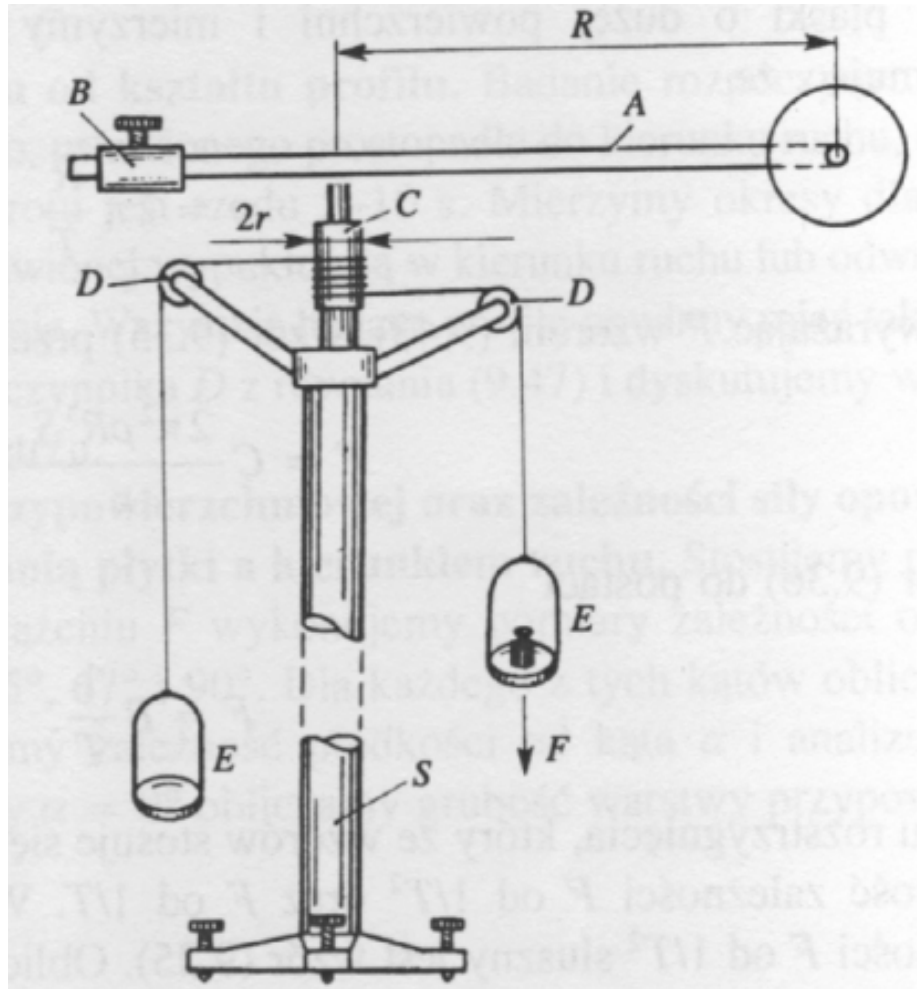
$S$  – is the cross-section area of the object measured in perpendicular to the direction of its movement,

$\rho$  - the fluid density (in our experiment it is the air density of  $\rho = 1.2928 \text{ kg/m}^3$ )

$v$  - the object velocity

$C$  - proportionality coefficient.

For a given object and fluid, it is possible to check which formula better describes the drag force dependence on the object velocity on the basis of measurements with the use of aerodynamic carousel.



The main element of the instrument is a heavy stand made of metal, S. On top of it there is the rotating arm A and at the end of this arm the profiles studied are mounted. The profile weight is balanced by a counterbalance B. The vertical rotating element below the arm is set in small friction bearings, and is inside a cylinder C on which there is a thread with the help of which the carousel is set in rotating motion. The thread is run through two blocks D and attached to two bowls E. When the bowls are not loaded the system is in equilibrium, by loading one of them the system is set in motion.

The force  $F$ , equal to the weight of the loading placed in a bowl is applied to the surface of the cylinder  $C$  of the radius  $r$  and the moment of the force is  $rF$ . The moment is transferred to the profile studied which is at a distance  $R$  from the axis of rotation, so that

$$rF = RP. \quad (2)$$

Find the aerodynamical drag coefficient  $C$  using (1), (2) and from formula for velocity in circular motion:

$$v = \frac{2\pi R}{T}. \quad (3)$$

We obtain:

$$F = C \frac{2\pi^2 S \rho R^3}{r} \frac{1}{T^2}, \quad (4)$$

and eventually, get:

$$T^2 = C \frac{2\pi^2 S \rho R^3}{rg} \frac{1}{m}. \quad (5)$$

Now, we can consider above equation as linear in slope-intercept form:  $y = ax$ . Here,  $y = T^2$ ,  $x = 1/m$  and slope of the line  $a = C \frac{2\pi^2 S \rho R^3}{rg}$ . From the slope of the line  $a$ , we can find  $C$ :

$$C = \frac{arg}{2\pi^2 S \rho R^3}. \quad (6)$$

Moreover, find a unit of measurement of coefficient  $C$  using following:

$$[F] = kg \frac{m}{s^2}, [S] = m^2, [R] = [r] = m, [T] = s \text{ and } [\rho] = \frac{kg}{m^3}.$$

Remember to measure parameters of the device ( $R$ ,  $r$ ), as well as tested objects in order to calculate surface  $S$ .

Calculate the uncertainty of  $C$  from:

$$\Delta C = \left( \left| \frac{\Delta a}{a} \right| + \left| \frac{\Delta r}{r} \right| + \left| -\frac{\Delta S}{S} \right| + \left| -3 \frac{\Delta R}{R} \right| \right) C. \quad (7)$$

Finally, compare drag coefficients for square and circular shapes:

$$|C_{circle} - C_{square}| \leq \Delta C_{circle} + \Delta C_{square}. \quad (8)$$

## Experiment F103

### Elastic collisions on the air track

Aim: Conservation of momentum

According to the law of momentum conservation if a system is isolated (i.e. not subjected to external forces) its total momentum does not change.

When two bodies endowed with momentums  $\vec{p}_1$  and  $\vec{p}_2$  collide, then according to the law of momentum conservation we have

$$\vec{p}_1 + \vec{p}_2 = \vec{p}'_1 + \vec{p}'_2,$$

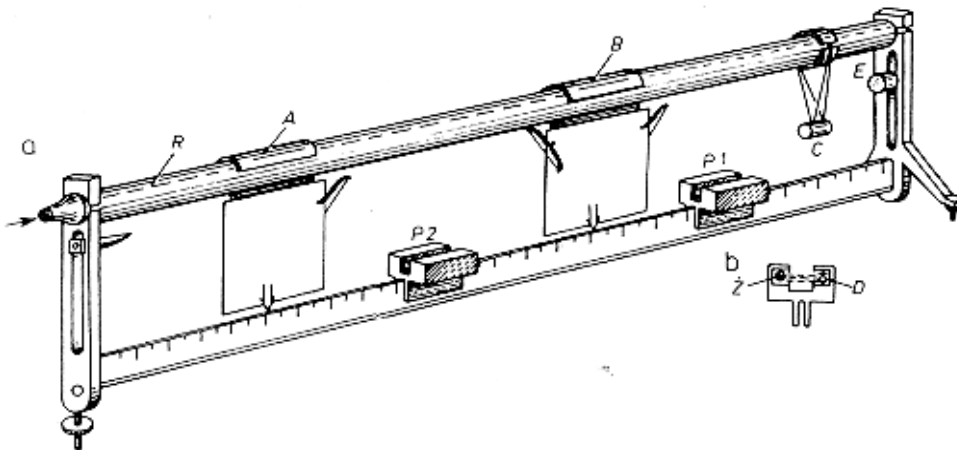
where

$\vec{p}'_1$  and  $\vec{p}'_2$  are the momentums after the collision.

Momentum is a vector and direction of the momentum vector  $\vec{p} = m\vec{V}$  is determined by the direction of velocity  $\vec{V}$ .

#### Experiment

The measurements are made on the air track whose construction permits the movements of cars with negligible friction.



#### Measurements

Clip car B having mass  $m_B$  to the electric fastening, then release the car and measure the time in which car B passes through gate P1. Repeat the measurement a few times. Calculate the initial velocity of car B that is  $V_B$  from the equation:

$$(1) \quad V_B = \frac{l_B}{t_B},$$

Where

$l_B$  is the length of car B.

Repeat the same procedure for car A and gate P2.

These data are needed to find the initial momentum.

At the next step collect the data for determination of the final momentum.

Let car B and car A collide. Measure the time  $t_A$  in which car A passes through gate P2 and calculate its velocity  $V_A$  from the formula analogous to eq. (1).

Measure the time  $t'_B$  in which car B passes through gate P1 after the collision with car A. The velocity of car B after the collision is  $V'_B$  :

$$(2) \quad V'_B = \frac{l_B}{t'_B}.$$

Using the electronic balance measure masses of cars A and B,  $m_A$  and  $m_B$ . Note the measurement uncertainties  $\Delta m_A$  and  $\Delta m_B$ .

Measure the lengths of cars A and B,  $l_A$  and  $l_B$ , note the uncertainties  $\Delta l_A$  and  $\Delta l_B$ .

Repeat many times (e.g. 20) the measurements of the times the cars pass the gates.

Calculate

a) the initial momentum using the simplified formula

$$(3) \quad p_p = p_A - p_B = m_A V_A - m_B V_B = m_A \frac{l_A}{t_A} - m_B \frac{l_B}{t_B}.$$

b) the measuring uncertainty of the initial momentum as

$$(4) \quad \Delta p_p = \frac{p_{p \max} - p_{p \min}}{2},$$

where

$$p_{p \max} = \frac{(m_A + \Delta m_A)(l_A + \Delta l_A)}{(t_A - \Delta t_A)} - \frac{(m_B - \Delta m_B)(l_B - \Delta l_B)}{(t_B + \Delta t_B)},$$

$$p_{p \min} = \frac{(m_A - \Delta m_A)(l_A - \Delta l_A)}{(t_A + \Delta t_A)} - \frac{(m_B + \Delta m_B)(l_B + \Delta l_B)}{(t_B - \Delta t_B)},$$

Or using the exact differential:

$$(5) \quad \Delta p_p = \left( \left| \frac{l_A}{t_A} \right| \Delta m_A + \left| \frac{m_A}{t_A} \right| \Delta l_A + \left| -\frac{m_A l_A}{t_A^2} \right| \Delta t_A \right) - \left( \left| \frac{l_B}{t_B} \right| \Delta m_B + \left| \frac{m_B}{t_B} \right| \Delta l_B + \left| -\frac{m_B l_B}{t_B^2} \right| \Delta t_B \right).$$

c) The final momentum is

$$(6) \quad p_k = -p'_A + p'_B = -m_A V'_A + m_B V'_B = -m_A \frac{l_A}{t'_A} + m_B \frac{l_B}{t'_B}.$$

d) The uncertainty of the final momentum calculated from the simplified formula is

$$(7) \quad \Delta p_k = \frac{p_{k \max} - p_{k \min}}{2},$$

where

$$p_{k \max} = -\frac{(m_A - \Delta m_A)(l_A - \Delta l_A)}{(t'_A + \Delta t'_A)} + \frac{(m_B + \Delta m_B)(l_B + \Delta l_B)}{(t'_B - \Delta t'_B)},$$

$$p_{k \min} = -\frac{(m_A + \Delta m_A)(l_A + \Delta l_A)}{(t'_A - \Delta t'_A)} + \frac{(m_B - \Delta m_B)(l_B - \Delta l_B)}{(t'_B + \Delta t'_B)}.$$

The uncertainty of the initial momentum can be found using the exact differential method and then it is

$$(5) \quad \Delta p_p = \left( \left| \frac{l_A}{t'_A} \right| \Delta m_A + \left| \frac{m_A}{t'_A} \right| \Delta l_A + \left| -\frac{m_A l_A}{t'^2_A} \right| \Delta t'_A \right) - \left( \left| \frac{l_B}{t'_B} \right| \Delta m_B + \left| \frac{m_B}{t'_B} \right| \Delta l_B + \left| -\frac{m_B l_B}{t'^2_B} \right| \Delta t'_B \right).$$

e) The final momentum can be found as

$$(6) \quad p_k = -p'_A + p'_B = -m_A V'_A + m_B V'_B = -m_A \frac{l_A}{t'_A} + m_B \frac{l_B}{t'_B}.$$

f) The uncertainty of the final momentum calculated using the simplified formula is

$$(7) \quad \Delta p_k = \frac{p_{k \max} - p_{k \min}}{2},$$

where

$$p_{k \max} = -\frac{(m_A - \Delta m_A)(l_A - \Delta l_A)}{(t'_A + \Delta t'_A)} + \frac{(m_B + \Delta m_B)(l_B + \Delta l_B)}{(t'_B - \Delta t'_B)},$$

$$p_{k \min} = -\frac{(m_A + \Delta m_A)(l_A + \Delta l_A)}{(t'_A - \Delta t'_A)} + \frac{(m_B - \Delta m_B)(l_B - \Delta l_B)}{(t'_B + \Delta t'_B)},$$

The uncertainty of the final momentum calculated using the exact differential method is

$$(8) \quad \Delta p_k = -\left( \left| \frac{l_A}{t'_A} \right| \Delta m_A + \left| \frac{m_A}{t'_A} \right| \Delta l_A + \left| -\frac{m_A l_A}{(t'_A)^2} \right| \Delta t'_A \right) + \left( \left| \frac{l_B}{t'_B} \right| \Delta m_B + \left| \frac{m_B}{t'_B} \right| \Delta l_B + \left| -\frac{m_B l_B}{(t'_B)^2} \right| \Delta t'_B \right).$$

g) The measuring uncertainties of times loaded with random uncertainty is calculated from the standard deviations

$$\Delta t_A = 3S_{t_A}, \quad \Delta t_B = 3S_{t_B},$$

$$\Delta t'_A = 3S_{t'_A}, \quad \Delta t'_B = 3S_{t'_B}.$$

In order to check if the law of conservation of momentum is satisfied, apply the following criterion of agreement:

$$(9) \quad |p_p - p_k| \leq \Delta p_p + \Delta p_k.$$

If the above inequality is met, you have shown that the law of conservation of momentum is valid. If not, analyze the reasons and identify the possible sources of error.

Knowing the relation between the momentum and kinetic energy of an object in motion:

$$(10) \quad E^k = \frac{p^2}{2m},$$

You can verify the law of energy conservation using your measurements.

$$(11) \quad |E_p^k - E_k^k| \leq \Delta E_p^k + \Delta E_k^k ?$$

For this purpose calculate

a) The initial kinetic energy as:  $E_p^k = \frac{p_A^2}{2m_A} + \frac{p_B^2}{2m_B}$

b) Uncertainty of the initial kinetic energy can be calculated by two methods, similarly as that of momentum,

- using the simplified formula as  $\Delta E_p^k = \frac{E_{p_{\max}}^k - E_{p_{\min}}^k}{2},$

where

$$E_{p_{\max}}^k = \frac{(p_A + \Delta p_A)^2}{2(m_A - \Delta m)} + \frac{(p_B + \Delta p_B)^2}{2(m_B - \Delta m_B)}, \quad E_{p_{\min}}^k = \frac{(p_A - \Delta p_A)^2}{2(m_A + \Delta m)} + \frac{(p_B - \Delta p_B)^2}{2(m_B + \Delta m_B)}.$$

- or using the exact differential as:

$$\Delta E_p^k = \left( \left| \frac{p_A}{m_A} \right| \Delta p_A + \left| -\frac{p_A^2}{2m_A^2} \right| \Delta m_A \right) + \left( \left| \frac{p_B}{m_B} \right| \Delta p_B + \left| -\frac{p_B^2}{2m_B^2} \right| \Delta m_B \right).$$



c) The final kinetic energy is  $E_k^k = \frac{p_A'^2}{2m_A} + \frac{p_B'^2}{2m_B}$

d) The uncertainty of the final kinetic energy can be calculated by two methods,

- using the simplified formula as  $\Delta E_k^k = \frac{E_{k \max}^k - E_{k \min}^k}{2}$ ,

Where 
$$E_{k \max}^k = \frac{(p_A' + \Delta p_A')^2}{2(m_A - \Delta m)} + \frac{(p_B' + \Delta p_B')^2}{2(m_B - \Delta m)}, \quad E_{k \min}^k = \frac{(p_A' - \Delta p_A')^2}{2(m_A + \Delta m)} + \frac{(p_B' - \Delta p_B')^2}{2(m_B + \Delta m)},$$

and  $p'$  stands for the final momentum,

- or by the exact differential:

$$\Delta E_k^k = \left( \left| \frac{p_A'}{m_A} \right| \Delta p_A' + \left| -\frac{p_A'^2}{2m_A^2} \right| \Delta m_A \right) + \left( \left| \frac{p_B'}{m_B} \right| \Delta p_B' + \left| -\frac{p_B'^2}{2m_B^2} \right| \Delta m_B \right).$$

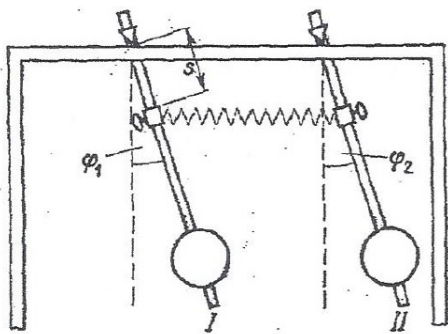
## Experiment F109.

### Oscillations of coupled pendulums

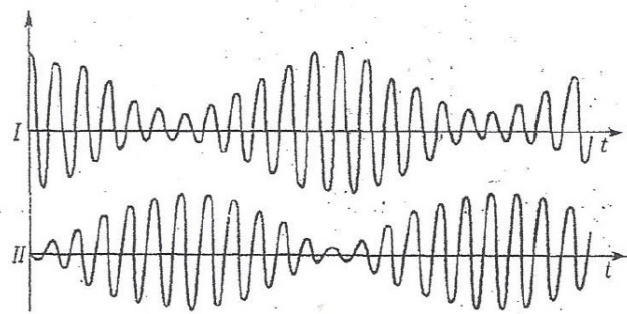
Aim: To verify if the frequency of beats is equal to the difference in the frequencies of normal oscillations

Coupled pendulums are two physical pendulums coupled with each other, which means connected through a device transmitting energy from one pendulum to the other. Mathematically oscillations performed by each pendulum can be described as superposition of two harmonic oscillations of little different frequencies, taking place along the same line. The component frequencies are called normal frequencies.

When two pendulums of close periods of normal oscillations are coupled the system shows beats. Let's assume that at the initial moment pendulum II is motionless, while pendulum I performs oscillations. Then the amplitude of oscillations of pendulum I decreases, while the amplitude of oscillations of pendulum II increases reaching a maximum when the amplitude of pendulum I decreases to zero, then the energy is passed to pendulum I. The time in which the amplitude of oscillations of one of the pendulums reaches the initial value is called the period of beats  $T_d$ , while the corresponding frequency  $\omega_d$  – is the frequency of beats.



A system of coupled pendulums



Oscillations of coupled pendulums - beats

No transmission of energy through oscillations is observed if the pendulums perform the so-called normal oscillations. The first normal oscillation takes place when the pendulums perform oscillations in the same phase:  $\varphi_1 = \varphi_2$ , these oscillations are characterized by the angular frequency  $\omega_1$ , known as the first normal frequency and corresponds to the period  $T_1$ . The second normal oscillation takes place when the pendulums perform oscillations in the opposite phases:  $\varphi_1 = -\varphi_2$ , with the second normal angular frequency  $\omega_2$  and period  $T_2$ .

The first and second angular frequencies meet the relations:

$$\omega_2 = \omega_1 = \omega_d [1]$$

Measurements and statistics of results

	$t_1 [ \quad ]$	$t_{11} [ \quad ]$
1		
2		
3		
4		
5		

1. Repeat 5 times the measurement of the period of the first and second oscillations .
2. Repeat 5 times the measurement of the periods of the first and second normal oscillations.

	$t_1 [ \quad ]$ $\varphi_1 = \varphi_2$	$t_2 [ \quad ]$ $\varphi_1 = -\varphi_2$
1		
2		
3		
4		
5		

3. Repeat 5 times the measurement of the periods of beats.

	$t_d [ \quad ]$
1	
2	
3	
4	
5	

4. The formulae for estimation of uncertainties of direct measurements

$$\Delta_d t =$$

$$S_{\bar{t}} = \sqrt{\frac{1}{n(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2}$$

$$\Delta_c t = \Delta_d t + 3S_{\bar{t}}$$

5. Determine the frequencies of normal vibrations  $\omega_1$  and  $\omega_2$  for pendulums I and II on the basis of measurements made.

$$\bar{t}_I =$$

$$S_{\bar{t}_I} =$$

$$\Delta_c t_I =$$

$$T_I = \frac{t_I}{5} =$$

$$\omega_I = \frac{2\pi}{T_I} =$$

$$\Delta_c \omega_I = \omega_I \frac{\Delta_c t_I}{t_I} =$$

$$\bar{t}_{II} =$$

$$S_{\bar{t}_{II}} =$$

$$\Delta_c t_{II} =$$

$$T_{II} = \frac{t_{II}}{5} =$$

$$\omega_{II} = \frac{2\pi}{T_{II}} =$$

$$\Delta_c \omega_{II} = \omega_{II} \frac{\Delta_c t_{II}}{t_{II}} =$$

6. Determine the frequencies of normal oscillations  $\omega_1$  and  $\omega_2$  on the basis of the measurements:

$$\bar{t}_1 =$$

$$S_{\bar{t}_1} =$$

$$\Delta_c t_1 =$$

$$T_1 = \frac{t_1}{5} =$$

$$\omega_1 = \frac{2\pi}{T_1} =$$

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7. Find the frequency of beats  $\omega_d$  on the basis of the measurements :

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$$=$$

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$$=$$

$$\Delta_c \omega_d^{(1)} = \frac{\omega_{d2 \max} - \omega_{d2 \min}}{2} =$$

9. Final results

$$\bar{t}_I \pm \Delta_c t_I =$$

$$\bar{t}_{II} \pm \Delta_c t_{II} =$$

$$\bar{t}_1 \pm \Delta_c t_1 =$$

$$\bar{t}_2 \pm \Delta_c t_2 =$$

$$\bar{t}_d \pm \Delta_c t_d =$$

$$\omega_I \pm \Delta_c \omega_I =$$

$$\omega_{II} \pm \Delta_c \omega_{II} =$$

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$$\omega_d \pm \Delta_c \omega_d =$$

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10. Check if the formula [1] is valid:

$$\left| \omega_d - \omega_d^{(1)} \right| \leq \Delta_c \omega_d + \Delta_c \omega_d^{(1)}$$

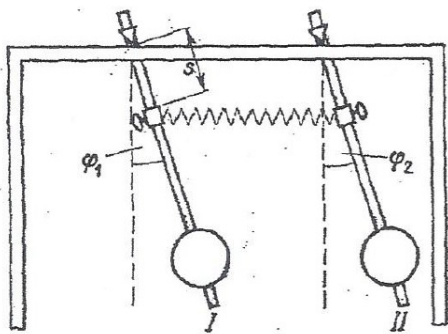
## Experiment F109.

### Oscillations of coupled pendulums

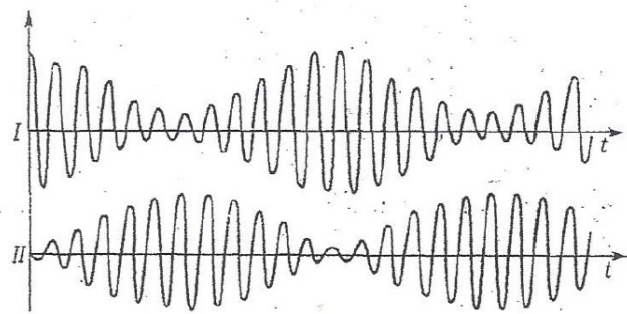
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$$\Delta_c \omega_1 = \omega_1 \frac{\Delta_c t_1}{t_1} =$$

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$$S_{\bar{t}_2} =$$

$$\Delta_c t_2 =$$

$$T_2 = \frac{t_2}{5} =$$

$$\omega_2 = \frac{2\pi}{T_2} =$$

$$\Delta_c \omega_2 = \omega_1 \frac{\Delta_c t_2}{t_2} =$$

7. Find the frequency of beats  $\omega_d$  on the basis of the measurements :

$$\bar{t}_d =$$

$$S_{\bar{t}_d} =$$

$$\Delta_c t_d =$$

$$T_d = \frac{t_d}{5} =$$

$$\omega_d = \frac{2\pi}{T_d} =$$

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10. Check if the formula [1] is valid:

$$\left| \omega_d - \omega_d^{(1)} \right| \leq \Delta_c \omega_d + \Delta_c \omega_d^{(1)}$$

## M102. Transformations of energy in the motion of a pendulum

### Aim:

- Determination of the real motion of a model of mathematical pendulum
- Measurement of time dependence of the pendulum position  $\alpha(t)$ ,
- On the basis of  $\alpha(t)$  calculation of time dependencies of the pendulum velocity  $v(t)$ , kinetic energy  $E_k(t)$ , potential energy  $E_p(t)$ , total energy  $E_c(t)$ ,
- Measurement of the period  $T$  of pendulum oscillations as a function of the amplitude of oscillations  $\alpha_0$ ,
- Numerical calculation of the dependence  $T(\alpha_0)$  and its comparison with experimental one.

### Problems:

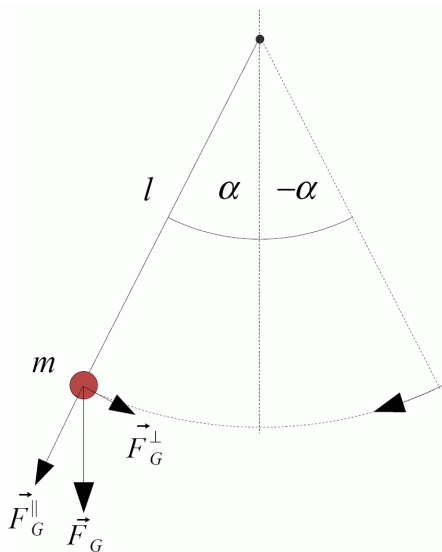
- Harmonic motion,
- Equation of the mathematical pendulum in the approximation of small angles,
- Newton's second law of dynamics.

### References:

- Henryk Szydłowski "Pracownia Fizyczna", Wyd. Nauk. PWN, Warszawa 1994
- David Halliday, Robert Resnick, Jearl Walker "Podstawy fizyki" Wyd. Nauk. PWN, Warszawa 2003

## 1. Introduction

Mathematical pendulum (Fig. 1) is a mass  $m$  concentrated at a point and hanged on a weightless string of the length  $l$ , performing periodical oscillations about the point of equilibrium. The oscillations are performed without air resistance and without friction at the point of attachment. Such a pendulum is an idealised representation of a real physical pendulum.



**Figure 1.** Model of a mathematical pendulum with the forces acting on the mass.

Analysis of the pendulum oscillations is based on the Newton's second law of dynamics. The driving force is the component  $\vec{F}_G^{\perp}$  of the gravitation force  $\vec{F}_G$ , tangent to the arc over which

the mass  $m$  moves. The value of the force is given by

$$\vec{F}_G^\perp = -F_G \sin(\alpha) = -mg \sin(\alpha), \quad (1)$$

where  $g$  is the gravitational constant. According to the Newton's second law of dynamics, the force is known to produce a motion with instantaneous acceleration  $a$  meeting the relation,

$$\vec{F}_G^\perp = ma. \quad (2)$$

Taking into account the direction of the force, the kinematic definition of acceleration  $a = d^2s/dt^2$  (where  $s$  is the displacement) and the relation between the length of the arc,  $s$ , and the amplitude of the pendulum,  $\alpha$ ,  $s = l\alpha$ , we finally have,

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l} \sin(\alpha) = 0. \quad (3)$$

Equation (3) does not have a simple analytical solution. Only on condition that the amplitude of the pendulum is small so that the relation  $\sin(\alpha) = \alpha$  is satisfied, we get a well-known in physics equation for the harmonic oscillations

$$\frac{d^2\alpha}{dt^2} + \frac{g}{l} \alpha = 0. \quad (4)$$

whose solution is,

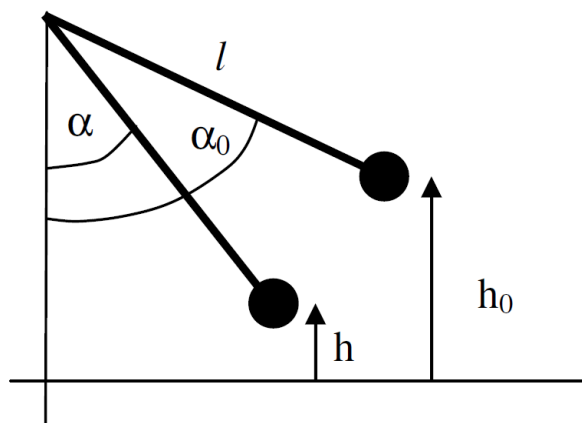
$$\alpha(t) = \alpha_0 \cos(\omega_0 t + \delta) \quad (5)$$

where  $\alpha_0$  is the amplitude of oscillations,  $\omega_0 = \sqrt{g/l}$  is the frequency of normal oscillations of the pendulum and  $\delta$  is the initial phase of the movement for  $t = 0$ . Knowing the frequency of oscillations it is possible to calculate the period of a pendulum in the approximation of small amplitude, from the following equation,

$$T = 2\pi\sqrt{l/g} \quad (6)$$

In general, replacement of eq.(3) by eq.(4) is not justified, which means that the pendulum oscillations are not harmonic and the period of pendulum motion is a function of amplitude.

A simple numerical integration permits finding the period of oscillations for a specific amplitude  $\alpha_0$ .



**Figure 2.** Illustration for numerical determination of the pendulum period.

Increase in the kinetic energy at the transition from  $\alpha_0$  to  $\alpha$  is equal to the potential energy difference at these positions,  $mg(h_0 - h)$ . Assuming  $\alpha_0$  as the starting point ( $v = 0$ ), we have

$$\frac{mv^2(\alpha)}{2} = mg(h_0 - h) \quad (7)$$

And, because  $h(\alpha) = l[1 - \cos(\alpha)]$ , where  $l$  is the length of the pendulum,

$$\frac{mv^2(\alpha)}{2} = mgl(\cos(\alpha) - \cos(\alpha_0)) \quad (8)$$

thus



$$v(\alpha) = \sqrt{2gl(\cos(\alpha) - \cos(\alpha_0))} \quad (9)$$

The time needed for covering the arc section  $ds = l d\alpha$  is  $dt = ds/v$ . Integration of  $dt$  over the angle  $\alpha_0$  to 0 gives one fourth of the pendulum period:

$$\frac{T}{4} = \int_{\alpha_0}^0 \frac{l d\alpha}{\sqrt{2gl(\cos(\alpha) - \cos(\alpha_0))}} \quad (10)$$

Simple procedures for numerical integration are given below.

The aim of this experiment is to determine the dependence of the period of the pendulum oscillations,  $T$ , on their amplitude,  $\alpha_0$ . Numerical integration of eq. (10) permits the calculation of theoretical value of the pendulum period and its dependence on the oscillations amplitude. Thus, the experimentally determined  $T(\alpha_0)$  dependence can be compared with the simulated one. On the basis of numerical calculations you will be able to find the instantaneous values of velocity, kinetic energy, potential energy and total energy.

In our experiment, the mathematical pendulum is approximated by a relatively heavy cylinder (~0.5kg) set on a light aluminium rod of about 1 m in length. Large inertia of the pendulum, its hanging on a high quality ball bearing and small mechanical resistance of the potentiometer used as a sensor of position make such a system a reasonable model of a mathematical pendulum.

### Tasks

- Measure the length of the pendulum and its mass.
- Calculate the period of pendulum from formula (6).

## 2. Measuring setup.

The quantity to be directly measured is voltage  $U$  measured on a potentiometer whose axis of rotation coincides with the axis of the pendulum rotation. In order to determine the real angle of pendulum deviation from the position of equilibrium, corresponding to the voltage  $U$ , you should measure the voltage at the position of equilibrium ( $U_0$ ) and the voltage  $U_\beta$  for a large known amplitude  $\beta$ , e.g. when the pendulum reaches the restricting boundary. Then, the voltage measured  $U$  can be easily converted into the amplitude  $\alpha$ :

$$\alpha = \beta \frac{U - U_0}{U_\beta - U_0} \quad (11)$$

The voltage is measured with the use of an analog-digital converter card connected to a computer. The communication between the computer and the card is realized through the available set of subroutines.

### 3. Solution of differential equations of motion(numerical integration)

The problem is to solve the equation of motion on the basis of the known acting force. We want to know current values of the velocity and position of the pendulum on the basis of the instantaneous acceleration calculated from the Newton's second law of dynamics, as the ratio of the instantaneous force to mass.

$$f(t) \approx \frac{F(t + \Delta t) - F(t)}{\Delta t} \quad (12)$$

$$F(t + \Delta t) \approx F(t) + f(t)\Delta t \quad (13)$$

Following the procedure described by equations (12) and (13), you should calculate the position  $x$  as a function of time at first for the pair acceleration ( $f=a$ ) and velocity ( $F=v$ ) and then for the pair velocity ( $f=v$ ), and position ( $F=x$ ). By choosing a suitably small value of  $\Delta t$ , it is possible to get a good approximation of the real movement of the pendulum. It should be remembered that the driving force of the pendulum is also a function of time, through the dependence on the angle  $\alpha$ , eq.(1).

#### Tasks

Write a program in LabView to simulate the motion of a mathematical pendulum. As initial parameters use  $l, g, \Delta t$  and  $\alpha_0$ . You should get as a result a quantity of Waveform type containing the information on the pendulum positions at subsequent moments of time  $t_i = i\Delta t$ . When writing the equations remember about the signs representing the sense of particular quantities, e.g. acceleration is always directed in the opposite to the direction of deviation from the position of equilibrium. .

### 4. Numerical differentiation

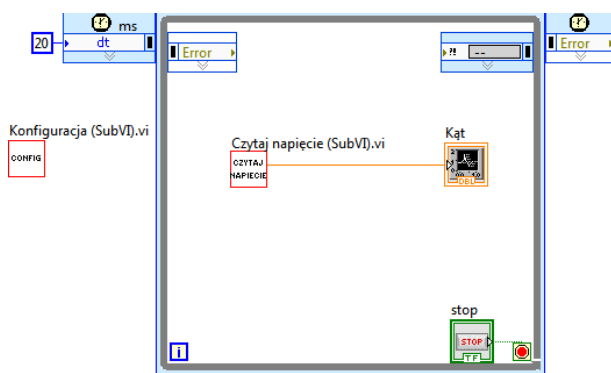
Use again the approximation described by eq. (12), but this time the function  $F(t)$  is known and  $f(t)$  should be found directly from this equation. In this experiment numerical differentiation is used for determination of instantaneous values of the pendulum velocity.

### 5. Experiment

## Write program or programs according to the scheme given below.

The subVI needed for communication with the measuring device are given in the palette "Functions->User Libraries->Pendulum".

1. Write a program for continuous reading the voltage drop on the potentiometer mounted at the pendulum rod attachment. Use subVI "Configure (subVI)" and "Read voltage (SubVI)". Use the "Timed Loop" While to perform synchronous voltage reading at the rate of 50 values per second (delay of 20ms). The values measured display in the WaveForm Chart



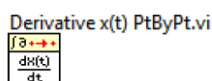
2. Use the measured values of voltage and convert them into the angle, voltage [V] into angle [rad]. Using the program written, measure the voltage  $U_0$  corresponding to the equilibrium position of the pendulum. This value corresponds to  $\alpha=0$ .
  - b) Measure the voltage corresponding to the pendulum deviation by 90 to the left ( $U_{+90}$ ) and to the right ( $U_{-90}$ ), for the range of 180 stopni= $\pi$ . Find the calibration constant from the equation  $C[\text{rad}/\text{V}] = \frac{\pi}{|U_{+90} - U_{-90}|}$
3. Use the function "Normalise (SubVI)" to convert continuously the values of voltage into angles of amplitude. Send the value  $U_0$  to the connector "Zero shift[V]", while the value  $C$  to the connector "Scaling constant[rad/V]".

## Mechanical energy relations in pendulum motion

1. Make the function (SubVI), to calculate the potential energy of the pendulum on the basis of read off values of the amplitude angles  $E_p = mgL(1 - \cos(\alpha))$
2. Make the function (SubVI), to calculate the kinetic energy of the pendulum on the basis of the current values of angular velocity,

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\omega^2 = \frac{1}{2}mL^2\left(\frac{d\alpha}{dt}\right)^2$$

To calculate the derivative use the function “Derivative x(t) PtByPt.vi” to calculate the derivative point by point.



Besides the current value of the amplitude angle, to calculate  $d\alpha$ , the function needs the value of  $dt$ . When calculating the derivative numerically,  $dt$  is  $\Delta t$  so the time distance between two measurements of the angle, in the example presented 20ms)

3. Find out the total energy as a sum of kinetic and potential energy terms. Draw the time dependencies of all three terms. Analyse the relationship between the energies, potential energy is exchanged into kinetic energy. The total energy is constant for small time intervals. Over longer time intervals you will note the effect of damping. Over a longer time interval the effect of damping is well seen. This effect is related to the energy dissipation in the friction of the bearing, the potentiometer and the resistance of air. Try to guess the character of the time dependence of the energy dissipation, is it linear or perhaps exponential?

## Dependence of the pendulum period on its amplitude – $T(\alpha_0)$

1. Record the time changes in the pendulum amplitude, starting from high amplitudes to almost total disappearance of oscillations. Write the values of times and angles in a table and then present them as a plot of  $\alpha(t)$  in XY Graph.
2. Take advantage of XY Graph cursors to find out the period of oscillations for a given amplitude (angle). In order to do this, find the positions of the minima or maxima of amplitude.
3. Write a new program (VI). Write the read off cursor positions (the values of  $T$  and  $\alpha_0$ ) into the one-dimensional tables. Make an XY Graph of  $T(\alpha_0)$ . Comment on the result obtained taking

into account its comparison with the result obtained for a mathematical

pendulum ( $T(\alpha_0) = \text{const.}$ ),  $T = 2\pi \sqrt{l/g}$ .

4. Estimate the value of period that would be observed for  $\alpha_0 \rightarrow 0$ . Using this value and the equation for the pendulum period, calculate the length of the pendulum. Compare the value obtained with the real length of the pendulum. Comment on the result, answer which value is higher and why.

## Additional tasks

1. Read the additional materials provided.
2. Determine the relation  $T(\alpha_0)$  from numerical integration of full (not simplified) equation of motion (3). Compare the results with experimental data.
3. Estimate the error following from the approximation of the real pendulum by a mathematical pendulum. Calculate the period of the real pendulum treating it as a physical pendulum. For the sake of simplicity assume that the weight has a ball shape. Weight the aluminium rod.

## Technical remarks

### Numerical integration

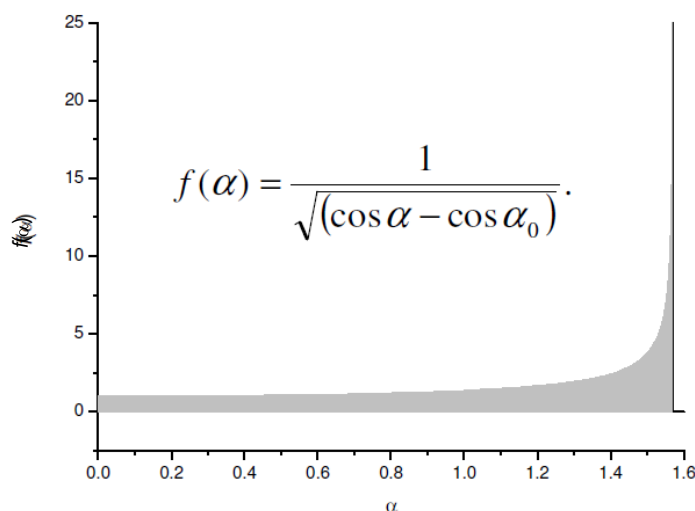
*The content given below is given just for illustration. The methods described do not ensure the optimum precision or rate of execution. Those interested should consult any handbook on numerical methods. .*

#### a) Definite integral in finite limits. The method of trapeziums.

The geometrical interpretation of a definite integral  $F = \int_a^b f(x) dx$  is the area under the function to be integrated. The interval of integration  $\langle a, b \rangle$  is divided into  $n$  parts of the same size, at the points  $x_1, x_2, x_3, \dots, x_{n+1}$  of the width  $\Delta x = (b-a)/n$ . The area  $P_i$  under the plot of the function  $f(x)$  in the section  $(x_i, x_{i+1})$  can be approximated by the area of a trapezium of the sides  $f(x_i)$  and  $f(x_{i+1})$  and the height of  $\Delta x$ , so  $P_i = [f(x_i) + f(x_{i+1})] \Delta x / 2$ . This implies the following equation

$$F = \left( f(a) + f(b) + 2 \sum_{i=1}^{n-1} f(x_i) \right) \frac{\Delta x}{2}.$$

In LabView you can use the function “Numerical Integration”, whose arguments are the values of the integrated function given in a table and the interval  $\Delta x$ . The interval  $\Delta x$  should be chosen taking into account the rate of the function changes. It is recommended to continue the calculations for decreasing value of  $\Delta x$  as long as it stops showing significant changes.



**Figure 3.** The plot of a fragment of the integrand from eq. (10). The integral is equal to the area marked in grey.

Unfortunately, the integrand from eq. (10) is discontinuous for  $\alpha = \alpha_0$  (see Fig. 3). As the integral is finite despite this, the period of the pendulum has a defined value, the solution is to reduce the value of  $\Delta x$  and approaching  $\alpha_0$  to such a point that the integral value stops changing significantly. As no increase in the density of division is needed in the range through which the integrand is relatively flat, it is recommended to divide the integral into a sum of two integrals calculated for different  $\Delta x$ , different density of division. As the point of division we suggest 90 % of the range of integration. It can be mathematically presented as follows:

$$F = \sum_{j=1}^N F_j = \sum_{j=1}^N \int_{a_j}^{b_j} f(\alpha) d\alpha$$

where

$$a_1 = a, \quad b_j = b - \frac{b-a}{10^j}, \quad a_{j+1} = b_j.$$

For illustration we give a few values of  $a_j, b_j$  for  $a = 0$  and  $b = 1$ :

$j$	1	2	3	4	5	6
$a_j$	0	0.9	0.99	0.999	0.9999	0.99999
$b_j$	0.9	0.99	0.999	0.9999	0.99999	0.999999

Assuming a division of each section into  $n = 100$  points, we should get a good accuracy of the

integral for  $N=8$ , so after  $8 \times 10^7 = 8000$  steps. Note that for the density of division from the last section extended over the entire range of integration it is necessary to perform  $10^8$  steps.

### *Tasks*

*Write a program in LabView to calculate the period of a mathematical pendulum as a function of amplitude of oscillations. After selection of the optimum values of  $n$  and  $N$  express it in the form of sub-Vi with inputs  $l$ ,  $g$ ,  $\alpha_0$  and output giving  $T$ .*

## M104. Free fall and sliding on the ramp

### Aims:

- *Observation of electromotive force induction in a coil by a magnet moving inside the coil.*
- *Observation of changes in the position and velocity of a body performing a free fall and sliding over a ramp.*
- *Measurement of the acceleration of a body sliding over a ramp as a function of the ramp inclination angle.*
- *Determination of the friction coefficient characterising the friction between the sliding body and ramp surface.*

### 1. Introduction

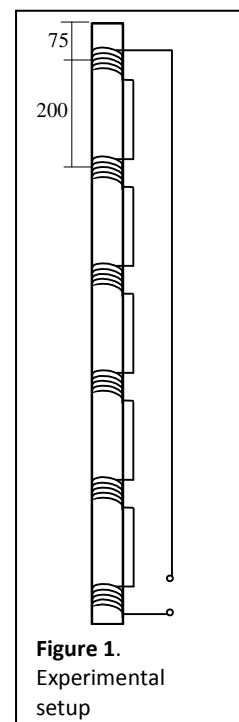
Galileo Galilei is regarded as one of the fathers of contemporary science in recognition of his studies in the area of astronomy and physics. One of his greatest achievements was performance of the accurate effect of the gravitational force on the free fall of bodies. Galileo assumed that upon free fall the velocity of the falling body should increase by the same value in the same time so at a constant rate. In his times the testing of this hypothesis was a great challenge. Galileo was not able to measure the motion of a free falling body in time as there was no technology allowing recording of so fast changes. To overcome this problem Galileo decided to decrease the velocity of the falling body by placing it on a ramp. This solution is justified by the fact that the free fall can be treated as sliding on a vertical ramp. Thus the body sliding over an inclined ramp should change its velocity in the same way as a freely falling body.

### 2. Equipment

A plastic tube with a system of equidistant coils, permanent magnets.

### 3. Instruments

Changes in velocity of a given object when it is moving can be realised by measuring time changes in the object position, so measuring the path dependence on time. Determination of a moving object position needs a possibility of checking its position over a relatively long path. Such a possibility can be realised using a plastic tube of 1.15 m in length on which 6 equidistant and connected in series coils are wound up. The first coil is wound up at a distance of 7.5 cm from the upper end of the tube. The other coils are wound at every 20





cm. Each coil is about 1 cm long, has the internal diameter of 1.5 cm, while the external diameter of 1.6 cm. The coils are identical and are made of 17 loops of copper wire of 0.5 mm in diameter. The ends of the wire are connected to the sockets through which they are linked to the measuring interface. This construction permits the use of electromagnetic induction (IEM) to determine the moment of time at which a given object passes the coil, although the falling object has to be a magnet. In this experiment two permanent magnets differing in the type of surface, will be used. Their use will permit determination of the effect of a static friction (between the surface of the magnet and the surface of the ramp) on the movement of the object.

## 4. Theoretical introduction

### *Electromagnetic induction*

Because it is related to the method for the measurement of position of a sliding body, let's refresh your knowledge on electromagnetic induction.

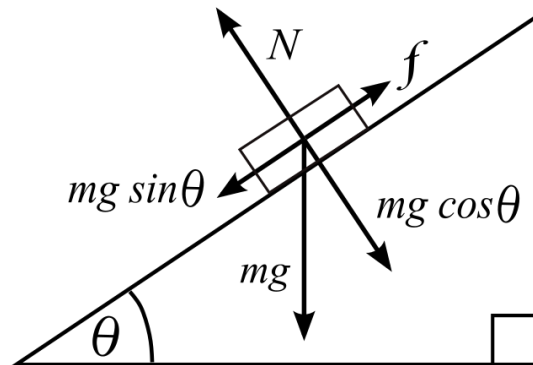
In short, the electromagnetic induction is the phenomenon of generation of an electromotive force  $E$  in an electric circuit upon time changes in the flux of magnetic induction passing through the circuit. The source of a magnetic field can be a permanent magnet. The electromotive force generated is proportional to the rate of changes in the magnetic flux  $\Phi$ .

$$E = - \frac{d\Phi}{dt}. \quad (1)$$

The minus sign means that the direction of the induced current is such that the effects of the current oppose the phenomenon (the change) generating it, the Lenz rule. When a permanent magnet passes through a cell of  $N$  wire loops, the total value of  $E$  can be approximated by the equation

$$E = -N \frac{d\Phi}{dt}. \quad (2)$$

Measurement of  $E$  as a function of time permits determination of certain parameters related to the magnetic field of the falling magnet.

***A body on the ramp***

The figure presents a body of mass  $m$  placed on a ramp inclined at the angle  $\theta$  to the flat surface and the forces acting on it.  $N$  is the force of pressure,  $g$  is the earth gravitational constant,  $f$  is the friction force,  $f = \mu N$ , where  $\mu$  is the coefficient describing the static friction between the surface of the body and the surface of the ramp. Analysis of the forces presented in the figure leads to the well-known equation describing the acceleration of the sliding body as a function of the ramp inclination angle,

$$a = g(\sin(\theta) - \mu \cos(\theta)). \quad (3)$$

The acceleration calculated from this equation can be compared with that measured in the experiment in order to find out the static friction coefficient.

**5. Realisation of the experiment****A. Observation of electromotive force induction**

1. Start the program “Spadek.vi” [Fall.vi] and record the time course of voltage induced in the coil by the magnet moving inside it.
2. Note that the voltage induced in each subsequent coil changes linearly with the time of the magnet fall. What does this observation imply about the magnet velocity?

**B. Free fall of a body**

1. Start the program “Spadek.vi” [Fall.vi] and similarly as above record the time changes in the induced voltage. Press the button “Zakończ pomiar i przejdź do analizy” [Stop the measurement and go to analysis].
2. In the tab “Analiza” [Analysis] you will find the results of analysis of the measured time

changes in the voltage induced; moment of the start of motion, acceleration determined from the time dependence of the path for the minima and maxima. The tables also give the positions and values of the minima and maxima of the measured induced voltage.

3. Change the angle of the ramp inclination to get the maximum value of the measured acceleration,  $a$ . This value corresponds to the acceleration of the body sliding on the vertical ramp,  $\theta=90^\circ$ . Make the measurements for two magnets differing in the material on their surfaces. Why the accelerations measured for the two magnets are different although the tube is in the vertical position? Why the acceleration values measured differ from the expected value equal to the gravitational constant  $g$ ?
4. Perform a few measurements and write the acceleration values measured for the two magnets.

### C. Friction in sliding motion

1. Using the program "spadek.vi" [Fall.vi] measure the accelerations of the magnets  $a$ , as a function of the ramp inclination angle,  $\theta$ . Reduce the ramp inclination by the step of  $5^\circ$  starting from  $90^\circ$ .
2. Continue the measurements until the body on the ramp will stop sliding.
3. Draw the values of  $a/g$  as a function of the ramp inclination angle  $\theta$  (assume that  $g=a$ ,  $\theta=90^\circ$ ).
4. Compare the measured relation  $a/g(\theta)$  with the relation given by eq. (3). Change the values of static friction coefficient,  $\mu$ , as long as it takes to get the theoretical relation closest to the experimental data, (the quality of the fit evaluate subjectively).
5. Are you able to linearize equation (3) to be able to apply linear regression analysis for determination of  $\mu$ ?
6. Calculate the static friction coefficients,  $\mu$ , for the two magnets. Calculate the critical angles.

## D103. Coupled pendulums – small angle approximation

*Aim: Examination of oscillations of two coupled physical pendulums*

- *examination of relation between the frequency of oscillations of a simple pendulum and its moment of inertia, determination of the centre of gravity.*
- *examination of motion of two identical pendulums coupled through a spring*
- *examination of two different pendulums coupled through a spring.*

**Problems:**

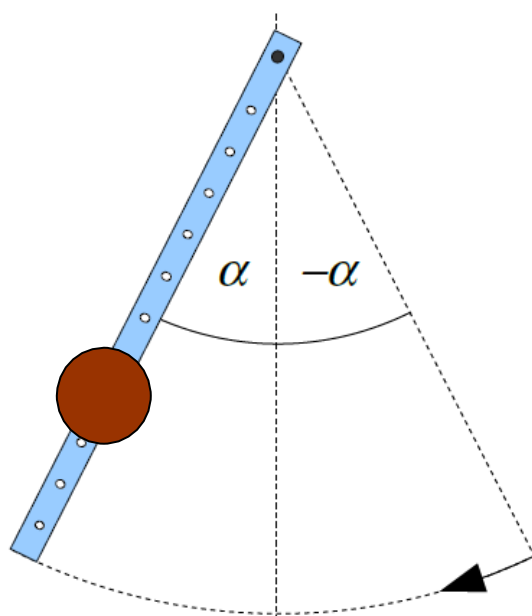
- *equation of motion for a physical pendulum, frequency of free oscillations, relation between the moment of inertia of a body and the frequency of its free oscillations.*
- *normal oscillations, equations describing normal oscillations for a system of two coupled pendulums, sum of normal oscillations, beats.*

**Experimental tools needed:**

- *two physical pendulums to which weights can be attached. A spring for coupling the two pendulums.*
- *potentiometers connected to each pendulum at the pivot points, that return voltage as a function of angular position of each pendulum.*
- *interface permitting measurement of voltage on the potentiometer by a computer.*
- *LabView environment through which it is possible to connect with the measuring interface and perform measurements. The voltage at particular potentiometers is read via the function `Odczyt.vi` or in the more advanced form – via the serial communication with the measuring interface.*

### 1. Free oscillations of physical pendulum.

Most often studied is a mathematical pendulum which is a simplified version of a physical one in which the bob performing oscillations is assumed to be a point of mass and the rod (or cord) on which the bob swings is assumed to have no mass. In this experiment the physical pendulums are used, see Fig. 1.



**Figure.1.** A physical pendulum used in the experiment.

The difference between a physical and mathematical pendulums is that the physical one is a rigid body able to rotate about the pivot point which is other than the centre of gravity of this body. To describe its motion it is necessary to take into account the moment of inertia  $I$  of the pendulum and position of its centre of gravity with respect to the pivot point. The Newton's second law of motion for rotations takes the form:

$$M = \frac{d\vec{L}}{dt} \quad (1)$$

where  $M$  is the net moment of force acting on the pendulum, while  $L$  is the moment of momentum related to the moment of force applied to the pendulum.  $L$  is related to the moment of inertia  $I$  of the rotating body through the equation:

$$\vec{L} = I\vec{\omega} \quad (2)$$

where  $\omega$  is the vector of angular velocity with the same direction and sense as  $M$ . The above equations can be applied to physical as well as mathematical pendulums. The difference appears in determination

of  $M$ . For both types of pendulum, the only source of the moment of force is the gravity. For the mathematical pendulum the moment of force is:

$$|\vec{M}| = |\vec{r} \times \vec{F}_G| = mgl \sin(-\alpha) = -mgl \sin(\alpha) \quad (3)$$

where  $m$  is the mass of the body and  $l$  is the distance from the pivot point or the length of the pendulum. The sign minus means that, similarly as for a spring, the moment of force acting on the pendulum and related to gravity is directed so that to counteract the displacements and return the pendulum towards the equilibrium position.

In the physical pendulum, the gravitation force acting on each fragment of the rigid body is the same, while the distances between each fragment and the pivot point are different. Therefore, the moment of force acting on the pendulum must be determined as a net moment of all moments acting on infinitesimally small fragments  $m_i$  of the body. The net moment of force is equal to the sum of all component moments

$$|\vec{M}_g| = \int |d\vec{M}_i| = \int g r_i \sin(-\alpha) dm_i = -g \sin \alpha \int r_i dm_i = -D \sin \alpha \quad (4)$$

where  $r_i$  is the distance of mass  $dm_i$  from the pivot point,  $D$  is the directing moment of the pendulum.

Taking into account that  $\omega = \frac{d\alpha}{dt}$ , and assuming small amplitude  $\alpha$ , which permits assuming that  $\sin(\alpha) \approx \alpha$ , and combining equations (1), (2) and (4) we get the equation of motion for a single physical pendulum:

$$\frac{d^2\alpha}{dt^2} + \frac{D}{I}\alpha = 0 \quad (5)$$

Please note that the amplitude is the only parameter varying in time that is needed for description of motion of a given pendulum. The solution to the differential equation (5) takes the form:

$$\alpha = \alpha_0 \cos(\omega_0 t + \varphi) \quad (6)$$

where  $\omega_0 = \sqrt{\frac{D_p}{I}}$  is the frequency of free oscillations of the pendulum,  $\alpha_0$  is the amplitude that can be found from the angular frequency ( $\frac{d\alpha}{dt}$ ) after having assumed appropriate initial conditions (note that angular frequency differs from the pendulum eigenfrequency).  $\varphi$  is the initial phase of the oscillation.

Determination of the moment of inertia of the pendulum which is a rod with holes, presented in Fig. 1, is not an easy task, however, in approximation we can neglect the holes and find  $I$  for a rectangular rod of the dimensions  $a$ ,  $b$  and  $c$ . Moment of inertia  $I_s$  of such a rod is measured with respect to the axis of rotation passing through its centre of mass and parallel to the side  $c$  of the rod and can be found from the equation:

$$I_s = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_0^c (x^2 + y^2) \rho dx dy dz = \frac{1}{12} m_p (a^2 + b^2) \quad (7)$$

where  $\rho$  is the density of the material of the rod. In our experiment the rod oscillates about the pivot point at a distance  $r$  from the centre of the rod mass. The centre of mass is in the middle of the uniform rod, so  $r$  can be measured and using the Steiner theorem the moment of inertia of the rod  $I_p$  with respect to the axis passing through the pivot point (black dot in Fig. 1).

$$I_p = I_s + mr^2 \quad (8)$$

A cylinder weight can be attached to the rod at different sites. If it is attached, then also its moment of inertia should be added to the total moment of inertia of the pendulum,

$$I = I_p + I_w, \text{ gdzie} \quad (9)$$

$$I_w = \frac{1}{2} m_w R_w^2 + m_w d_w^2.$$

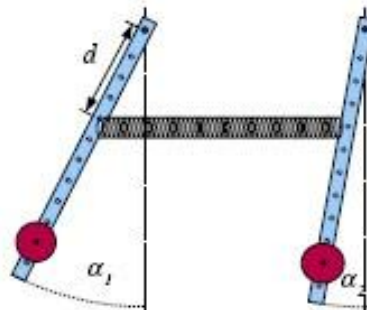
In the above equation for the total moment of inertia  $I$  of the rod with weight,  $m_w$  is the mass of the cylinder,  $R_w$  is its radius, and  $d_w$  is the distance between the pivot point and the centre of mass of the cylinder.

### Tasks

- Using the program provided, record the time dependence of the amplitude of one of the pendulums with the cylinder weight at the bottom end of the rod. Set the pendulum in motion with the maximum possible amplitude and let it stop. With the help of cursors read off the frequency  $f$  of the pendulum oscillations at different amplitudes  $\alpha_0$  and make the plot of  $f(\alpha_0)$ . Find out at which amplitude  $\alpha_g$  the approximation of small amplitudes ensures the acceptable accuracy. The following measurements make at the selected or smaller amplitude  $\alpha_g$ .
- Changing the positions of the cylinders on the rods, on one rod from bottom up and on the other from top down, find the frequencies of free oscillations of the pendulums as a function of  $d_w$ .
- Measure the dimensions of the rods, positions of the holes in which cylinders can be attached and the dimensions and mass of the cylinder weights. Knowing that the directing moment of the pendulum with a cylinder weight  $D$  is a sum of the directing moments of the rod and the cylinder, calculate the frequencies of free oscillations of the pendulum for three different positions of the weight and compare the results with experimental data.

## 2. Normal oscillations.

If the two pendulums are connected by a spring, in the way presented in Fig. 2, we get a system of two coupled pendulums.



**Figure 2.** A system of two coupled pendulums.

This system has two degrees of freedom which are the amplitudes of the two pendulums  $\alpha_1$  and  $\alpha_2$  (in Fig. 2 the angles have positive signs according to the convention assumed in Fig. 1). It should be remembered that the spring must be attached in such a way that it would be in equilibrium for the two pendulums in equilibrium. The system is called a sympathetic pendulum and it behaves in the same way as a system of two balls connected through a spring with each other and with the neighbouring walls, which is the simplest model of two atoms linked by a bond inside a molecule or a crystal. Although the motion of a sympathetic pendulum is complex, it can always be described as a superposition of two



independent harmonic motions (because of the two degrees of freedom). These motions, known as the normal oscillations, are independent and involve simultaneous oscillations of both pendulums in one or the other type of motion. The normal oscillation is the one in which all coordinates (in our experiment  $\alpha_1$  and  $\alpha_2$ ) change with the same frequency and with the same or exactly opposite phase. The equation describing the normal oscillations is:

$$\alpha_1 = A \cos(\omega t + \delta), \quad \alpha_2 = B \cos(\omega t + \delta). \quad (10)$$

The normal oscillations have strictly defined frequency and amplitude. Before we find them, we have to define the equation of motion for the sympathetic pendulum, which is different from eq. (5) because of the presence of a spring. In the system of pendulums from Fig. 2, besides the gravitational force also the restoring force, acting in the horizontal direction, contributes to the moment of force  $M$  (4). The force depends on the state of the spring (extension/ contraction  $\Delta x$ ) and the elasticity constant  $k$ ,  $\vec{F}_s = -k\Delta\vec{x}$ . Let's consider one of the two pendulums. A change in the length of the spring relative to its length at equilibrium depends on the amplitudes of the two pendulums in the following way:

$$\Delta x = d \sin \alpha_2 - d \sin \alpha_1 \quad (11)$$

The moment of the restoring force is:

$$M_s = dF_s \sin\left(\frac{\pi}{2} - \alpha_1\right) = -kd^2(\sin \alpha_2 - \sin \alpha_1) \cos \alpha_1 \quad (12)$$

and the moment of the gravitation force according to eq.(4) is

$$M_G = -mgr \sin \alpha_1 \quad (13)$$

The net moment of force is thus equal to the sum of  $M_S$  and  $M_G$ . Taking into account the approximation of small amplitude,  $\sin(\alpha_1) \approx \alpha_1, \sin(\alpha_2) \approx \alpha_2, \cos(\alpha_1) \approx 1, \cos(\alpha_2) \approx 1$ , we can derive the equations of motion for both pendulums, as we have done earlier. We get:

$$\frac{d^2\alpha_1}{dt^2} + (\omega_0^2 + H)\alpha_1 - H\alpha_2 = 0 \quad (14)$$

$$\frac{d^2\alpha_2}{dt^2} + (\omega_0^2 + H)\alpha_2 - H\alpha_1 = 0 \quad (15)$$

The frequency of free oscillations  $\omega_0$  is assumed to be the same (the pendulums are loaded with the same weights), while  $H = \frac{kd^2}{I}$ . Equations (14) and (15) describe the oscillations of both pendulums which can be decomposed into normal oscillations (10). In order to determine the frequencies and amplitudes of normal oscillations, the equation for  $\alpha_1$  and  $\alpha_2$  (10) is substituted to (14) and (15) to get the set of equations

$$\begin{aligned} \left[ (\omega_0^2 - \omega^2 + H)A - HB \right] \sin(\omega t + \delta) &= 0 \\ \left[ -HA + (\omega_0^2 - \omega^2 + H)B \right] \sin(\omega t + \delta) &= 0 \end{aligned} \quad (16)$$

This set of equations (16) must be satisfied for each moment, so

$$\begin{aligned} (\omega_0^2 - \omega^2 + H)A - HB &= 0 \\ -HA + (\omega_0^2 - \omega^2 + H)B &= 0 \end{aligned} \quad (17)$$

Set (17) has a non-trivial solution when the determinant of the matrix of coefficients is zero.

$$\begin{vmatrix} \omega_0^2 - \omega^2 + H & -H \\ -H & \omega_0^2 - \omega^2 + H \end{vmatrix} = 0 \quad (18)$$

Equation (18) leads to a quadratic equation:

$$\omega^4 - 2(\omega^2 + H)\omega^2 + \omega_0^4 + 4H\omega_0^2 = 0 \quad (19)$$

whose solutions are:

$$\omega_1 = \omega_0 \quad (20)$$

$$\omega_2 = \sqrt{\omega_0^2 + 2H} \quad (21)$$

Substituting the frequencies (20) and (21) to equations (17) we realise that

$$\text{for } \omega_1 = \omega_0 \quad B_1 = A_1, \quad (22)$$

$$\text{for } \omega_2 = \sqrt{\omega_0^2 + 2H} \quad B_2 = -A_2 \quad (23)$$

Equation (22) means that if the two pendulums oscillate in the same frequency equal to the frequency of free oscillations, then the phases of their motions are the same and the amplitudes of their oscillations are the same. Equation (23) describes the situation when the frequencies of the pendulums oscillations are different from that of free oscillations, amplitudes of the two pendulums are the same but the phases are the opposite. As mentioned before, any oscillations of coupled pendulums can be described as a superposition of normal oscillations. Using equations (10), (22) and (23) we get

$$\alpha_1 = A_1 \cos(\omega_1 t + \delta_1) + A_2 \cos(\omega_2 t + \delta_2) \quad (24)$$

$$\alpha_2 = A_1 \cos(\omega_1 t + \delta_1) - A_2 \cos(\omega_2 t + \delta_2) \quad (25)$$

The normal oscillations of pendulums can be obtained by choosing proper initial conditions. For the normal oscillations type I, eq. (22), the two pendulums should be set in motion with the same initial amplitudes. For the normal type II, eq.(23), the two pendulums should be set in motion with the same initial amplitudes but in the opposite directions.

**Tasks:**

- *Using the available program, measure the time dependencies of two coupled pendulums of known frequencies of free oscillations, moments of inertia and directing moment of the pendulum, for a few different coupling constants determined by the distance between the spring and the axis of rotation. .*
- *Read off the frequencies of normal oscillations of the sympathetic pendulum and compare with the calculated values.*
- *In order to do this you have to determine experimentally the elasticity constant of the spring,  $k$ . You can do this with the use of a weight of a known mass, thread or a piece of wire for fastening of the weight to the spring and a ruler.*

## 4. Beats .

Any oscillation can be described knowing its initial conditions. Let's assume that only one pendulum is set in motion with the initial amplitude of  $\alpha_1(0) = \alpha_0$ . Let at  $t=0$ , the initial phase be  $\delta_1 = \delta_2 = 0$ ,  $\alpha_2=0$ , and the initial velocities of the two pendulums be zero,

$\frac{d\alpha_1}{dt} \Big|_{(t=0)} = 0$ ,  $\frac{d\alpha_2}{dt} \Big|_{(t=0)} = 0$ . Substituting these values to equations (24) and (25) gives:

$$\alpha_1(0) = \alpha_0 = A_1 + A_2, \quad (26)$$

$$\alpha_2(0) = 0 = A_1 - A_2, \quad (27)$$

Having added or subtracted equations (26) and (27) by sides, we get  $A_1 = A_2 = \frac{\alpha_0}{2}$ . Finally, inset these values to equations (24) and (25), which gives the following relations:

$$\alpha_1 = \frac{\alpha_0}{2} (\cos \omega_1 t + \cos \omega_2 t) = \alpha_0 \cos \frac{\omega_1 - \omega_2}{2} \cos \frac{\omega_1 + \omega_2}{2} \quad (28)$$

$$\alpha_2 = \frac{\alpha_0}{2} (\cos \omega_1 t - \cos \omega_2 t) = -\alpha_0 \sin \frac{\omega_1 - \omega_2}{2} t \sin \frac{\omega_1 + \omega_2}{2} t \quad (29)$$

Substituting

$$\frac{\omega_1 + \omega_2}{2} = \omega_{sr}, \quad \frac{\omega_1 - \omega_2}{2} = \omega_{mod} \quad (30)$$

we get

$$\alpha_1 = \alpha_0 \cos(\omega_{mod} t) \cos(\omega_{sr} t) = A_{mod}(t) \cos(\omega_{sr} t), \quad (31)$$

The two pendulums oscillate with the same average frequency  $\omega_{sr}$ , and their amplitudes are modulated with the frequency  $\omega_{mod}$ , and their phases are the opposite. This phenomenon is called beats. The single cycle in which the maximum amplitude of one pendulum is passed to the other pendulum which goes from zero amplitude to its maximum amplitude and then the energy is transferred back to the first pendulum until it reaches the maximum amplitude again, is called a single beat. The time period of completion of this cycle is called the period of beats and its reciprocal is the frequency of beats.

### **Tasks**

- *Using the available program, measure the time dependence of beats a system of two coupled pendulums after one of them has been put in motion.*
- *With the help of cursors, find the frequency of beats and the average frequency of oscillations of the two pendulums.*
- *Set the system of pendulums in motion so that they would perform the first and then the second normal oscillations. Record the time dependencies of the amplitudes of the two pendulums. With the use of the available program find the frequency of each normal oscillation.*
- *Calculate the frequency of beats and the average frequency of oscillations of both pendulums using the measured frequencies of both normal modes. Compare the results.*

### **5. List of references.**

1. Frank S. Crawford Jr. "Fale", wyd.IIPWN, Warszawa1975 – available at the Faculty of Physics library.
2. Henryk Szydłowski "Pracownia Fizyczna", Wyd.Nauk. PWN, Warszawa1994 – – available at the Faculty of Physics library.
3. Mirosław Bylicki "Wahadła sprzężone"-  
[http://www.phys.uni.torun.pl/~mirekb/ipf\\_zad\\_16.pdf](http://www.phys.uni.torun.pl/~mirekb/ipf_zad_16.pdf)
4. David Halliday, Robert Resnick I Jearl Walker "Podstawyfizyki"Wyd.Nauk. PWN, Warszawa 2003 – available at the Faculty of Physics library.

## D101. Properties of sound waves;

### Speed of sound and beats

**Aim:** To write a measuring application for investigation of selected properties of sound waves:

- measurement of speed of sound on the basis of the time of acoustic pulse propagation
- investigation of the wave interference (beats)

**Problems:**

- The wave equation, interrelations between the frequency  $f$ , angular frequency  $\omega$ , wavelength  $\lambda$ , wave velocity  $v$ , period  $T$ , wavenumber  $k$ , energy of acoustic wave, Fourier analysis.
- The formula for the sum of sines:  $\sin(\alpha + \beta)$ .

#### 1. Introduction

The equation describing the plane wave of frequency  $f$  and wavelength  $\lambda$ , propagating in the direction  $x$  takes the form:

$$y = A \sin(\omega t - kx), \quad (1)$$

where  $y$  is the deviation from the equilibrium position,  $A$  is the amplitude of deviation,  $\omega = 2\pi f$  is the angular frequency,  $k$  is the wavenumber ( $k = 2\pi/\lambda$ ). In reference to the sound wave,  $y$  has a sense of a change in acoustic pressure. In this experiment, the sources of sound are two column loudspeakers connected to the computer sound card of a computer. Acoustic signal is recorded via a usual microphone connected to the same sound card. As the sound wave is plane it does not undergo polarization.

Placing a microphone at a certain position ( $x = \text{constant}$ ), we get an acoustic signal in the sinusoidal form:

$$y = A \sin(\omega t - \varphi), \quad (2)$$

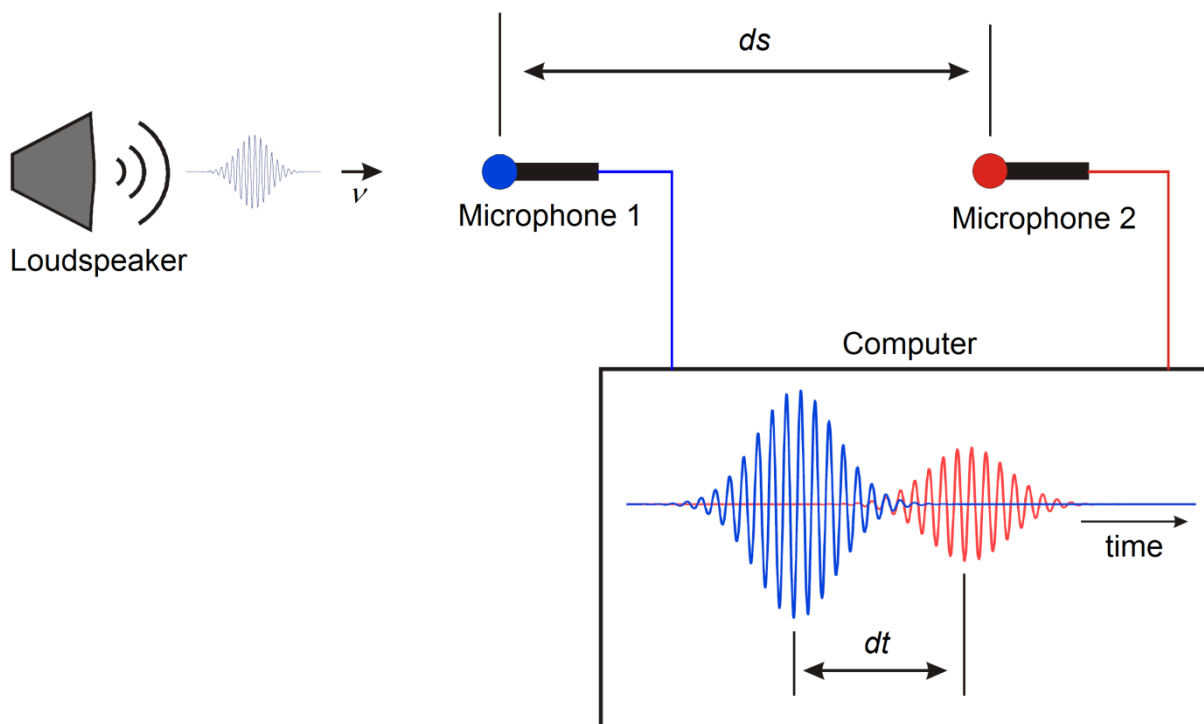
where  $\varphi = kx$ .

**Tasks:**

1. Write a program for generation of tones, based on the sound card and a set of loudspeakers. The frequencies and amplitudes should be controlled independently for the left and right channel.
2. Write a program for recording of sound by a microphone and sound card.
3. Record exemplary courses of acoustic waves coming from one source. Optimize the mutual positions of the loudspeakers and the microphone and optimize the settings in Properties of recording in the operation system. .
4. Comment: details on the sound generating and recording functions can be found in the section "Technical comments" given below. Start with tone generation, if you are satisfied with the result, add the function of recording. Both loops responsible for generation and recording of sound can be found in the same block diagram.

**2. Determination of sound speed from measurements of acoustic pulse propagation time**

The speed of sound can be determined by a more direct method. Assuming that sound propagates in air at the same speed, and making use of the definition of velocity in a uniform motion  $v=ds/dt$ , the speed of sound  $v$  can be found by generation of a short acoustic pulse (wave packet) and measurement of the time  $dt$  needed by the pulse to cover the distance  $ds$



A wave packet can be generated by multiplication of the sinusoidal wave by the bell-type function, e.g. a Gauss function,  $\exp[-((t - t_0)/\Delta t)^2]$ , where  $t_0$  is the position of the center and  $\Delta t$  is the width of the bell.

**Tasks**

1. Use one loudspeaker and two microphones set apart at a distance  $ds$ .
2. Modify the generator so that it would generate a tone of a given frequency whose amplitude is modulated by a Gauss function. In this way a pulse is obtained. The pulse duration is described by  $\Delta t$  (half width or full width at half maximum (FWHM)).
3. Record the acoustic signal from two microphones set at a distance  $ds$ . The image recorded should be made of two pulses whose maxima are shifted in time by  $dt$ . Use the function *Trigger and Gate.vi* to obtain a stable image of the pulses. Using the cursors, measure the time of pulse propagation  $dt$  in the *Waveform Graph*.
4. Measure the time of pulse propagation for a few distances between the microphones. Calculate the speed of sound from analysis of regression of the linear dependence  $ds(dt)$ .

**3. Investigation of wave interference**

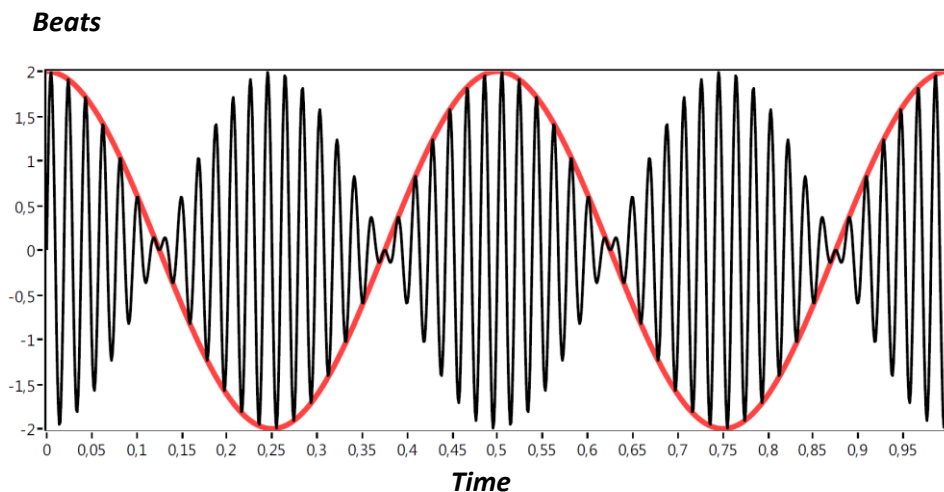
Waves coming from different sources, reaching simultaneously the same point in space undergo interference, which means that their momentary deviations. For two waves propagating in one direction, non-polarized or polarized in the same plane, we can write

$$y = A_1 \sin(\omega_1 t - k_1 x) + A_2 \sin(\omega_2 t - k_2 x) \tag{6}$$

If the frequencies of two interfering waves differ slightly, the phenomenon of beating takes place. This phenomenon heard as a modulation of the sound amplitude with the frequency equal to the difference between the frequencies of interfering waves. Neglecting the spatial term  $kx$ , we get:

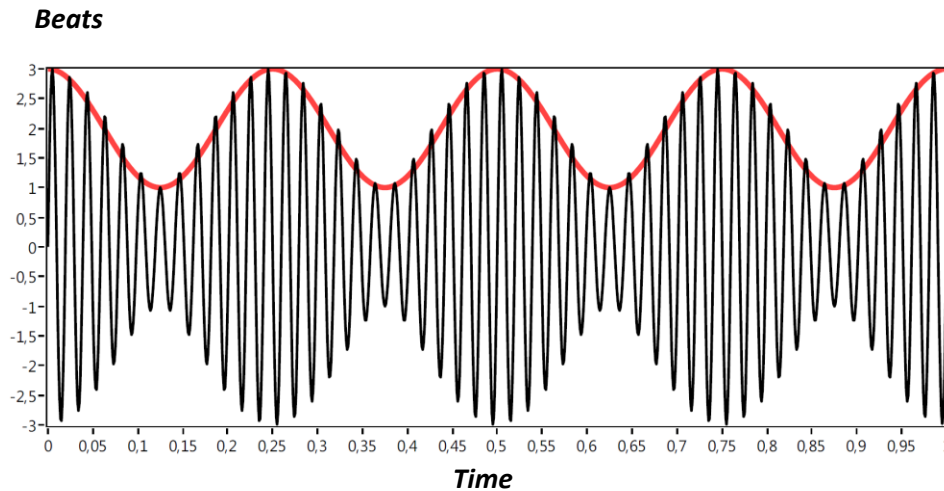
$$y = A[\sin(\omega_1 t) + \sin(\omega_2 t)] = 2A \cos[\frac{1}{2}(\omega_1 - \omega_2)t] \sin[\frac{1}{2}(\omega_1 + \omega_2)t], \tag{7}$$

in which the slowly oscillating term  $\cos[\frac{1}{2}(\omega_1 - \omega_2)t]$  can be treated as modulation of the amplitude of faster oscillations of the angular frequency  $\frac{1}{2}(\omega_1 + \omega_2)$ .





The depth of modulation decreases if the interfering waves differ in amplitude.



Although it is difficult to describe analytically the envelope of such beats, we can write the following equations:

$$\begin{aligned}
 y &= A_1 \sin(\omega_1 t) + A_2 \sin(\omega_2 t) \\
 &= A_1 \sin(\omega_1 t) + A_1 \sin(\omega_2 t) + (A_2 - A_1) \sin(\omega_2 t) \\
 &= 2A_1 \cos\left[\frac{1}{2}(\omega_1 - \omega_2)t\right] \sin\left[\frac{1}{2}(\omega_1 + \omega_2)t\right] + (A_2 - A_1) \sin(\omega_2 t).
 \end{aligned} \tag{8}$$

As follows, the interference of oscillations of similar frequencies can be expressed as deeply modulated oscillations of an intermediate frequency and oscillations of the component with highest amplitude. Unfortunately, this description does not fit the reality as the difference between the frequencies of these oscillations causes smoothing of the sharp fragments so that the envelope resembles a sinusoidal function of the frequency equal to the difference  $(\omega_1 - \omega_2)$ , and not to half of this difference.

Working on this

experiment we noted that the envelope of beats is rather well described by a combination of the two oscillating functions:

$$y = a \cos\left[\frac{1}{2}(\omega_1 + \omega_2)t\right] (b \cos[(\omega_1 - \omega_2)t] + c) \tag{9}$$

We leave the explanation of this observation to the inquiring students. In this experiment, you can determine the dependencies of parameters  $a$ ,  $b$ ,  $c$  on the ratio of amplitudes of the signals  $A_1/A_2$ . The values of these parameters you can get from manual fitting, e.g. by using slides in LabView.

Please note that beating is observed only when the waves can interfere on a detector. They are bound to occur if the source of sound are loudspeakers and a small microphone is a detector. For

comparison, try to listen to the same sound through the headphones (different frequencies in the left and right channels) or after shifting the loudspeakers directly to your ears (left to the left ear and right to the right one).

### **Task**

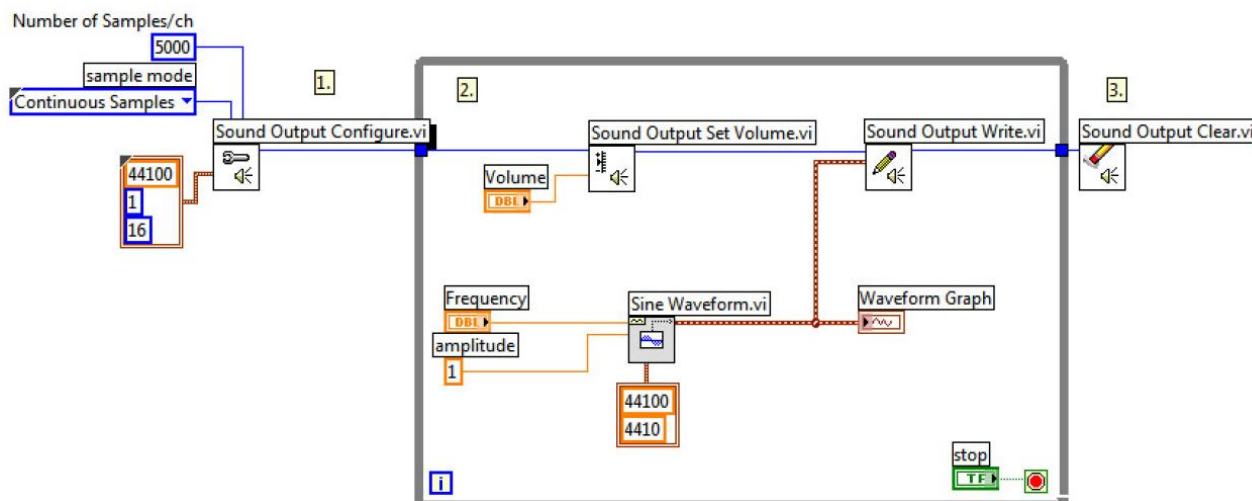
*Place two column loudspeakers next to each other and a microphone in front of them, at a distance of 20-40 cm. Generate the sound of slightly different frequencies in both loudspeakers. Measure the signal from the microphone for different combinations of frequencies and amplitudes of the signals from both loudspeakers. Describe the obtained envelope by equation (7) or (9). If you describe it by equation (9) try to interpret the parameters  $a$ ,  $b$ ,  $c$  in terms of the amplitudes of the component waves. The parameters of the component waves can be determined independently, by switching off individual columns.*

### **References**

1. Henryk Szydłowski “Pracownia Fizyczna”, Wyd. Nauk. PWN, Warszawa 1994.
2. David Halliday, Robert Resnick and Jearl Walker “Podstawy fizyki” Wyd. Nauk. PWN, Warszawa 2003.

## Technical comments:

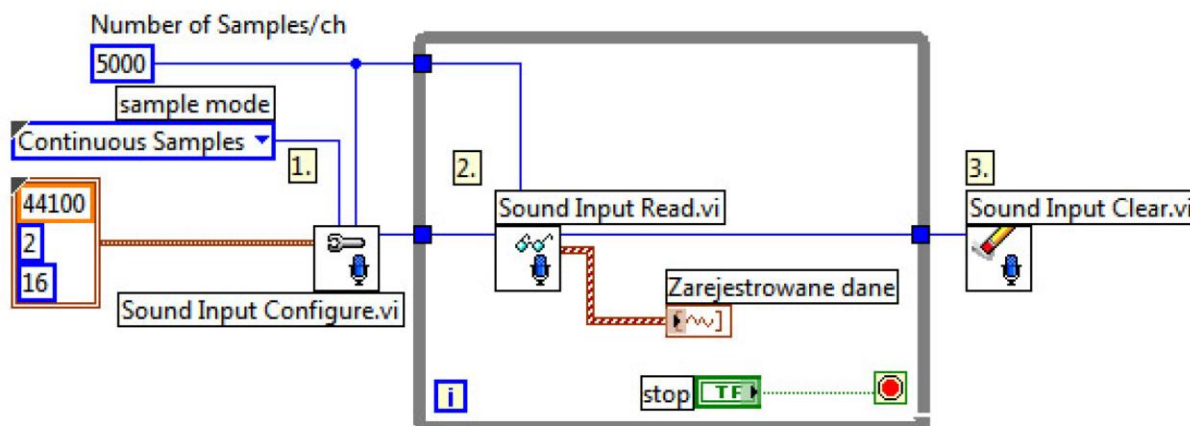
1. Because of the necessity of double channel (stereo) measurements in some measurements the microphones can be connected through the amplifier to the linear input of the sound card “line in” so that one of them works on the left channel, while the other with the right one. For the tasks for which one microphone is sufficient, it can be connected to the microphone input, checking the settings “Właściwości nagrywania” (Properties of recording), see point 2.
2. Check the settings “Właściwości nagrywania” (Properties of recording) in the system. Click twice the loudspeaker icon in the Windows tool bar. In the menu Options choose Properties and mark the volume of recording. Make sure that the box at linear input or microphone is marked, as needed. Set with a slide the volume of recording at least at the middle of the slide range.
3. A few comments on sound generation with the sound card
  - The functions permitting sound generation are to be chosen from the following Functions/Graphics & Sound/Sound/Sound Output LabView palette,
  - To generate a tone, it is preferable to use the function Sine Waveform.vi. Apart from the frequency and amplitude of sinus, the function needs the specification of the sampling rate and the number of samples of which the generated signal is to be composed. These parameters are needed to generate the waveform type data in LabView. It is a special data type which contains besides the samples table also information on the digital-to-analog converter setup. When using a music card, the sampling rate should be set to 44100 samples/second.
  - The diagram of the program given below generates a monophonic tone of the pre-set amplitude and frequency. For some experiments, e.g. investigation of beats, it is necessary to generate a stereo signal. To do it, introduce the appropriate change in the cluster Sound Format of the function Sound Output Configure. Use two functions Sine Waveform.vi. Then build the array from both waveforms (function Build Array) and send it to the function generating the Sound Output Write.vi.



1. Configure the device in continuous generation mode.
2. Repeat until user press Stop:
  - generate the sinus function (as a waveform) at set frequency and amplitude = 1. Show the data on the graph,
  - set the signal volume at the required level (0 - 100%),
  - play the waveform on the device by writing the data into the sound card.
3. Stop sound generation and clear all resources

#### 4. Comments on sound recording with the use of sound card

- The functions permitting sound generation are included in the following set: Functions/Graphics & Sound/Sound/Sound Input LabView palette,
- The diagram of the program given below generates the stereo signal and displays the data in a graph. The value written in the box *Number of Samples/ch* in the function *Sound Input Configure.vi* determines the number of samples of the signal so the time length of the signal recorded by *Sound Input Read.vi*. When you increase this value, the time length of signal recording will extend.



1. Configure the device in continuous registration mode.
2. Collect samples and show them in Graph until user press Stop.
3. Stop registration and clear all resources.

## 5. Modulation of tone amplitude; formation of a wavepacket.

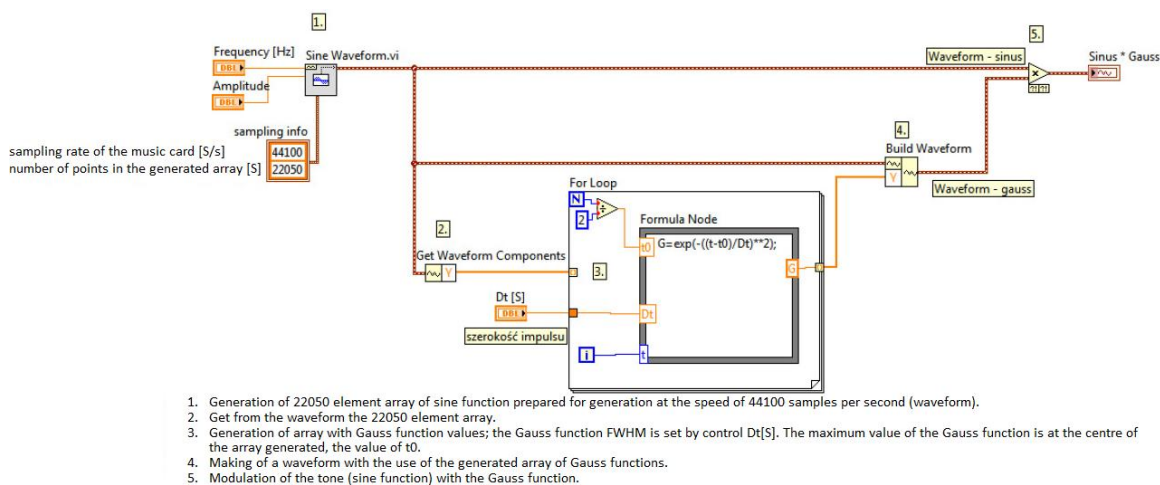
We are interested in generation of the function  $y(t) = A(t)T(t)$  where  $A(t)$  is the amplitude varying in time (Gauss function) and  $T(t)$  is the sine so the tone:

$$A(t) = e^{-[(t-t_0)/\Delta t]^2}$$

$$T(t) = \sin(2\pi ft)$$

To get properly generated modulated function, it must be expressed as a *Waveform*. Start with using the function *Sine Waveform.vi*. Use the function *Get Waveform Components* to collect from the waveform only the array of function values. Then, make the array of the values of Gauss function of the same size as the array of sines (loop *FOR* + structure of *Formula Node*). The array obtained use for building a waveform by using *Build Waveform*. The two waveforms for the Gauss function and for sinus function can be multiplied. The resultant waveform can be sent to the sound generating function *Sound Output Write.vi*.

The diagram of an exemplary VI realising the modulation of the sine function amplitude with a Gauss function is shown below.



6. When fitting a curve to the beats signal make use of the possibility of changing parameters  $a$ ,  $b$ ,  $c$  by slide controls or similar controls, as it permits fast choice of correct values of these parameters. Try to generate a complete function with a fast oscillating term, by setting its frequency to the value obtained from the measurements and by changing only the amplitude of this fast component.